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ON UNIQUE SOLVABILITY AND PICARD'S ITERATIVE METHOD FOR ABSOLUTE VALUE EQUATIONS

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Abstract

In this paper, we deal with unique solvability and numerical solution of absolute value equations (AVE), $Ax - B\,|x| = b$, $(A, B \in \mathbb{R}^{n\times n},\ b \in \mathbb{R}^n)$. Under some weaker conditions, a simple proof is given for unique solvability of AVE. Furthermore, we demonstrate with an example that these results are reliable to detect unique solvability of AVE. These results are also extended to unique solvability of standard and horizontal linear complementarity problems. Finally, we suggest a Picard iterative method to compute an approximated solution of some uniquely solvable AVE problems where its globally linear convergence is guaranteed via one of our weaker sufficient condition.

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1 Introduction

In this paper, we consider absolute value equations (abbreviated as AVE) of type:

$$Ax - B\,|x| = b, \tag{1}$$

where $A, B \in \mathbb{R}^{n\times n}$ are given, $b \in \mathbb{R}^n$, and $|x|$ is a vector whose i-th entry is the absolute value of the i-th entry of x. If $B = I$ the identity matrix, then the AVE (1) can be reduced to the type:

$$Ax - |x| = b. \tag{2}$$

Many problems in scientific and applied mathematics can lead to the solution of AVE. For example, boundary value problems [11], interval linear systems [6]

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and equilibrium problems [8]. Also the AVE includes the general NP-hard linear complementarity problem (LCP) which subsumes linear and convex quadratic programs [5]. The two research efforts in studying the AVE, one is purely theoretical analysis where researchers focus on the existence and uniqueness of solutions and the second is the numerical computation of solutions (cf. e.g. [3], [6], [8], [11]-[14]). For unique solvability of AVE, we cite the most well-known established results until today. In [10], Mangasarian and Meyer presented a sufficient condition, namely, $1 < \sigma_{\min}(A)$ for AVE (2). In [12], Rohn generalized this result to unique solvability of AVE (1) where he imposed the following sufficient condition

$$\sigma_{\max}(|B|) < \sigma_{\min}(A), \tag{3}$$

where $\sigma_{\max}(|B|)$ denotes the maximal singular value of matrix $|B| = (|b_{ij}|)$ and the $\sigma_{\min}(A)$ denotes the smallest singular values of matrix A. Furthermore, Lotfi and Veiseh [9], imposed other sufficient conditions that if the following matrix

$$A^T A - \||B|\|^2 I, \tag{4}$$

is positive definite, then AVE (1) is uniquely solvable for any $b \in \mathbb{R}^n$.

In this paper, we demonstrate that if matrices A and B satisfy either of the following conditions:

- $\sigma_{\min}(A) > \sigma_{\max}(B)$
- $\left\|A^{-1}B\right\| < 1$, provided A is non singular,
- the matrix $A^T A - \|B\|^2 I$ is positive definite,

then the AVE (1) is uniquely solvable for any b. The proof of our main results is based on the reformulation of the AVE (1) as a standard linear system of equations and then we show under our conditions that its matrix of coefficients is non singular. It is worth mentioning that our proof differs for the first condition from Rhon's proof [13], which is based on the alternative theorem and also for the second condition from the proof of Lotfi and Veiseh [9], which is inspired by the regularity of the interval matrix. Also it differs for the first condition from the proofs of Achache [1], Achache and Hazzam [3] and Wu and Li [15], where they involved the theory of linear complementarity problems with the class of P-matrix,i.e., a matrix with the determinants of all principal submatrices are positive (Theorem 3.3.7 in [5]). In addition, across an example of AVE (1), we demonstrate that our obtained results are reliable to detect unique solvability of AVE (1) rather than those stated in [9] and [12]. These results are also extended to standard LCP and its generalization horizontal LCP where through their reformulation as AVE (1), we deduce their unique solvability. Finally, a Picard iterative method is suggested to compute numerically an approximated solution for some uniquely solvable AVE problems. The globally linear convergence of the letter is guaranteed via the sufficient condition $\|A^{-1}B\| < 1$.

At the end of this section, some notations are presented. Let $\mathbb{R}^{n\times n}$ be the set of all $n\times n$ real matrices. The scalar product and the Euclidean norm are denoted, respectively, by $x^Ty, x, y \in \mathbb{R}^n$ and $\|x\| = \sqrt{x^Tx}$. The $sign(x)$ denotes a vector with the components equal to -1, 0 or 1 depending on whether the corresponding component is negative, zero or positive. In addition, $D(x) := Diag(sign(x))$ will denote a diagonal matrix corresponding to $sign(x)$. The absolute value of a matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ and the vector of all ones are denoted by $|A| = (|a_{ij}|) \in \mathbb{R}^{n\times n}$ and $e \in \mathbb{R}^n$, respectively. $\sigma_{\min}(A)$, $\sigma_{\max}(A)$ represent, respectively, the smallest and the largest singular value of matrix A. As is well known, $\sigma^2_{\min}(A) = \min_{\|x\|=1} x^TA^TAx$, and $\sigma^2_{\max}(A) = \max_{\|x\|=1} x^TA^TAx$. Finally, a matrix $A \in \mathbb{R}^{n\times n}$ is positive definite if for all nonzero vector x, $x^TAx > 0$ and the inverse of a non singular matrix A, is denoted by A^{-1}.

The remaining part of the paper is organized as follows. The main results are stated in section 2. In section 3, the obtained results are extended to standard and horizontal LCP. In section 4, a Picard's iterative method is suggested to provide an approximated solution for AVE (1). In section 5, some uniquely solvable AVE problems (including some standard and horizontal LCP problems) via our weaker sufficient conditions are detected. Moreover, the unique solution of these problems is computed via Picard's iterative method. A conclusion and some remarks are presented in section 6.

2 The main results

In this section, we will give our main results. First, for given matrices $A, B \in \mathbb{R}^{n\times n}$ and for any diagonal matrix $D \in \mathbb{R}^{n\times n}$ whose diagonal elements are ± 1 and 0, we define the matrix $(A - BD) \in \mathbb{R}^{n\times n}$. Then to achieve our main results, the following lemma is required.

Lemma 1. *Each of three conditions below implies the non singularity of* $(A-BD)$.

1. $\sigma_{\min}(A) > \sigma_{\max}(B)$,
2. $\left\|A^{-1}B\right\| < 1$, *provided A is non singular,*
3. *the matrix* $A^TA - \|B\|^2 I$ *is positive definite.*

Proof. For the first claim, assume that $(A - BD)$ is singular then,

$$(A - BD)x = 0, \text{for some}\, x \neq 0.$$

We then have

$$\begin{aligned}
\sigma^2_{\min}(A) = \quad & \min_{\|y\|=1} y^TA^TAy \leq x^TA^TAx = x^TDB^TBDx \\
\leq \quad & \max_{\|z\|=1} z^TDB^TBDz = \|BD\|^2 \\
\leq \quad & \|B\|^2 \|D\|^2 \leq \|B\|^2 = \max_{\|z\|=1} z^TB^TBz \\
= \quad & \sigma^2_{\max}(B)
\end{aligned}$$

which contradicts the first condition. Hence $(A - BD)$ is non singular. Next, by the same argument, assume that A is singular and let a nonzero vector x with $\|x\| = 1$ and such that

$$(A - BD)x = 0.$$

Next, because $x = A^{-1}BDx$, we then have,

$$\begin{aligned} 1 = \quad & \|x\| = \left\|A^{-1}BDx\right\| \\ \leq \quad & \left\|A^{-1}B\right\| \|D\| \|x\| \\ \leq \quad & \left\|A^{-1}B\right\|, \end{aligned}$$

which leads to a contradiction and hence $(A - BD)$ is non singular. For the last claim, assuming the contrary that $(A-BD)$ is singular, then for a nonzero vector x with $\|x\| = 1$, we have,

$$(A - BD)x = 0.$$

As $Ax = BDx$, we then have,

$$\begin{aligned} x^T A^T Ax - \|B\|^2 x^T x = \quad & x^T (BD)^T BDx - \|B\|^2 x^T x \\ = \quad & \|BDx\|^2 - \|B\|^2 x^T x \\ \leq \quad & \|B\|^2 \|D\|^2 \|x\|^2 - \|B\|^2 x^T x \\ \leq \quad & \|B\|^2 - \|B\|^2 = 0, \end{aligned}$$

and consequently

$$x^T A^T Ax - \|B\|^2 x^T x \leq 0.$$

This contradicts the fact that the matrix $A^T A - \|B\|^2 I$ is positive definite. Hence $(A - BD)$ is non singular for any diagonal matrix D whose elements are are ± 1 and 0. This completes the proof. $\square$

Next, according to $Dx = |x|$ where $D := Diag(sign(x))$, the AVE (1) can be rewritten [8] as the following standard linear system of equations:

$$(A - BD)x = b. \tag{5}$$

Then, it is clear that the AVE (1) is uniquely solvable for any b if the matrix of coefficients $(A - BD)$ of the linear system (5) is non singular for all diagonal matrix D whose diagonal elements are ± 1 or 0.

Theorem 1. *If matrices A and B satisfy*

1. $\sigma_{\min}(A) > \sigma_{\max}(B)$,
2. $\left\|A^{-1}B\right\| < 1$, *provided A is non singular,*
3. *the matrix $A^T A - \|B\|^2 I$ is positive definite,*

then the AVE (1) is uniquely solvable for any b.

Proof. Based on the results in Lemma 1, the matrix $(A - BD)$ of coefficients of the linear system (5) is non singular for any diagonal matrix D whose diagonal elements are ± 1 and 0. Hence the AVE (1) is uniquely solvable for any b. This completes the proof. $\square$

For AVE (2), the results of unique solvability are summarized in the following theorem.

Theorem 2. *If a matrix A satisfies either of the following conditions:*

1. $\sigma_{\min}(A) > 1$,
2. $\|A^{-1}\| < 1$, *provided A is non singular,*
3. *the matrix $A^T A - I$ is positive definite,*

then the AVE (2) has a unique solution for any b.

Proof. The proof is straightforwardly from Theorem 1, with $B = I$.

The following results concerning the unique solvability of the AVE (1) were provided in [9, 12].

Theorem 3 ([12])**.** *Let A and B satisfy*

$$\sigma_{\min}(A) > \sigma_{\max}(|B|),$$

then the AVE (1) has a unique solution for any b.

Theorem 4 ([9])**.** *Let $A, B \in \mathbb{R}^{n\times n}$ and the matrix*

$$A^T A - \||B|\|^2 I,$$

is positive definite, then the AVE (1) has a unique solution for any b.

Next example shows that Theorem 1 is reliable for detecting the unique solvability of AVE (1). Consider the following AVE (1) problem, where $A, B \in \mathbb{R}^{3\times 3}$ and $b \in \mathbb{R}^3$ are given by

$$A = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}, B = \begin{bmatrix} 4 & -2 & -2 \\ -2 & -5 & -2 \\ -2 & -2 & 2 \end{bmatrix}, b = \begin{bmatrix} 7 \\ 2 \\ 9 \end{bmatrix}.$$

By simple calculations, $\sigma_{\min}(A) = 7, \sigma_{\max}(B) = 6.1355$, $\sigma_{\max}(|B|) = 7.9018$. Theorem 3 [12], is not capable of detecting the unique solvability of the AVE since $\sigma_{\min}(A) = 6 < \sigma_{\max}(|B|) = 7.9018$. However, the application of Theorem 1 shows that $\sigma_{\min}(A) > \sigma_{\max}(B)$ which implies that the AVE has a unique solution for any $b \in \mathbb{R}^3$. Next, checking Theorem 4 [9], we have

$$A^T A - \||B|\|^2 I = \begin{bmatrix} -13.438 & 0 & 0 \\ 0 & -13.438 & 0 \\ 0 & 0 & -13.438 \end{bmatrix}.$$

So it is clear that $A^T A - \||B|\|^2 I$ is not positive definite. But Theorem 1, shows that

$$A^T A - \|B\|^2 I = \begin{bmatrix} 11.356 & 0 & 0 \\ 0 & 11.356 & 0 \\ 0 & 0 & 11.356 \end{bmatrix}$$

is positive definite and the AVE has a unique solution for any b. The unique solution of this example is $x^* = [1,\, -1,\, 1]^T$.

3 Unique solvability of standard and horizontal LCP

3.1 The standard LCP

In this section, our interest is to deduce sufficient conditions for the existence and uniqueness of the solution of standard LCP via those established in Theorem 1 for the AVE.

The linear complementarity problem [5], consists in finding a vector $x \in \mathbb{R}^n$ such that

$$y = Mx + q \geq 0,\, x \geq 0,\, x^T y = 0 \tag{6}$$

where $M \in \mathbb{R}^{n\times n}$ is a given matrix and $q \in \mathbb{R}^n$. Letting

$$y = |z| - z, \text{ and } x = |z| + z,$$

then the LCP (6) can be reformulated as the following AVE (1) as follows:

$$(I + M)z - (I - M)|z| = -q. \tag{7}$$

Then the LCP (6) has a unique solution for all q if and only if the AVE in (7) has a unique solution for any $b = -q$. Using Theorem 1 the following results of the unique solvability of the LCP are deduced.

Theorem 5. *Let $I + M$, $I - M \in \mathbb{R}^{n\times n}$ satisfy either of the following conditions:*
1- $\sigma_{\min}(I + M) > \sigma_{\max}(I - M)$,
2- $\|((I + M)^{-1}(I - M))\| < 1$, provided that the matrix $I + M$ is non singular,
3- $(I + M)^T(I + M) - \|I + M\|^2 I$ is positive definite,then the LCP has a unique solution for any q.

3.2 The horizontal LCP

The generalization of the standard LCP is called the horizontal linear complementarity problem (HLCP) [3, 5], consists also in finding a vector $x \in \mathbb{R}^n$ such that

$$Ny = Mx + q \geq 0,\, x \geq 0,\, x^T y = 0 \tag{8}$$

where $N, M \in \mathbb{R}^{n\times n}$ are given matrices and $q \in \mathbb{R}^n$. It is worth noting that the HLCP becomes a standard LCP if $N = I$, then the HLCP reduces to an LCP. By using the same change of variables, $y = |z| - z$ and $x = |z| + z$, the HLCP reduced to the following AVE (1)

$$(N + M)z - (N - M)|z| = -q. \tag{9}$$

According to this relation, then it is clear that the HLCP (8) has a unique solution for any q if and only if the AVE (9) is uniquely solvable for any $b = -q$. Again by Theorem 1, we provide the following results for the unique solvability of HLCP.

Theorem 6. *Let $N+M$, $N-M \in \mathbb{R}^{n\times n}$ satisfy either of the following conditions:*
1- $\sigma_{\min}(N+M) > \sigma_{\max}(N-M)$
2- $\|(N+M)^{-1}(N-M))\| < 1$*, provided that the matrix $(N+M)$ is non singular,*
3- $(N+M)^T(N+M) - \|N+M\|^2 I$ *is positive definite,*
then the HLCP has a unique solution for any q.

4 Picard's iterative method for AVE

In this section, in order to provide an approximated solution of some uniquely solvable AVE problems, a simple Picard's iterative method is proposed. First, we state the Banach fixed point theorem which will be used for proving the convergence of the proposed method, one can see [7] and [4] for its details proof.

Theorem 7. *(Banach's fixed point theorem). Let (X, d) be a non-empty complete metric space, $0 \leq \alpha < 1$ and $T : X \to X$ a mapping satisfying*

$$d(T(x), T(y)) \leq \alpha d(x, y), \text{ for all } x, y \in X.$$

Then there exists a unique $x \in X$ such that $T(x) = x$. Furthermore, x can be found as follows: start with an arbitrary element $x_0 \in X$ and define a sequence $\{x_k\}$ by

$$x_{k+1} = T(x_k),$$

then

$$\lim_{k\mapsto\infty} x_k = x$$

and the following inequalities hold:

$$d(x, x_{k+1}) \leq \frac{\alpha}{1-\alpha} d(x_{k+1}, x_k),\ d(x, x_{k+1}) \leq \alpha d(x, x_k).$$

Next, based on the fixed point principle, the sequence of iterations for solving the AVE (1) is given by

$$x_{k+1} = A^{-1}B\,|x_k| + A^{-1}b,\ k = 0, 1, 2, \ldots \tag{10}$$

Next under the condition 2 (Theorem 1), we provide a sufficient condition for the globally linear convergence of the fixed point iterations (10).

Theorem 8. *Let A be a non singular matrix and if*

$$\left\|A^{-1}B\right\| < 1$$

then the sequence $\{x_k\}$ converges to the unique solution x^* of the AVE (1) for any arbitrary $x_0 \in \mathbb{R}^n$. In this case the error bound is given by

$$\|x_{k+1} - x^*\| \leq \frac{\left\|A^{-1}B\right\|}{1 - \|A^{-1}B\|} \|x_k - x^*\|,\ k = 0, 1, 2, \ldots \tag{11}$$

Moreover, the sequence $\{x_k\}$ converges linearly to x^* as follows

$$\|x_{k+1} - x^*\| \leq \left\|A^{-1}B\right\| \|x_k - x^*\|,\ k = 0, 1, 2, \ldots \tag{12}$$

Proof. First, if the condition $\left\|A^{-1}B\right\| < 1$, holds then Theorem 2.1, implies that the AVE (1) is uniquely solvable for any b. Next, to prove the convergence of the sequence $\{x_k\}$ to x^*, we define the function $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\varphi(x) = A^{-1}B\,|x| + A^{-1}b.$$

Then it is easy to see with the help of the following inequality

$$\||x| - |y|\| \leq \|x - y\|, \text{ for all} x, y \in \mathbb{R}^n,$$

that

$$\|\varphi(x) - \varphi(y)\| \leq \left\|A^{-1}B\right\| \|x - y\|, \text{ for all } x, y \in \mathbb{R}^n.$$

Using Theorem 7 with $X = \mathbb{R}^n, T = \varphi, d(x, y) = \|x - y\|$ for all $x, y \in \mathbb{R}^n$ and $\alpha = \left\|A^{-1}B\right\| < 1$, we deduce the convergence of the sequence $\{x_k\}$ given by

$$x_{k+1} = \varphi(x_k),\ k = 0, 1, 2, \ldots$$

to the unique fixed point x^* to $\varphi(x)$ which is in turn the unique solution of the AVE (1). Moreover, the (11) and (12) hold which lead to the globally linear convergence of the method.

5 Checking unique solvability of AVE and numerical results

In this section, we present some examples of AVE problems including some examples of LCP where their unique solvability is checked. Also by applying Picard's iterative method, we compute an approximated solution of these examples. Our implementation is done by using the software Matlab. The starting point and the unique solution of the AVE are denoted, respectively, by x_0 and x^*. In the tables of numerical results we display the following notations: "Iter" and "CPU" state for the number of iterations and the elapsed times. The termination of the algorithm is as the relative residue:

$$\text{RSD} := \frac{\|Ax - B|x| - b\|}{\|b\|}$$

is less than the tolerance $\varepsilon = 10^{-6}$.

Example 1. Consider the problem of AVE where $A, B \in \mathbb{R}^{10\times 10}$ are given by:

$$A = \begin{bmatrix} 101 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 102 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 103 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 104 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 105 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & 106 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 107 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & 108 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 109 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 110 \end{bmatrix},\ B = I.$$

By a simple calculation, we get $\sigma_{\min}(A) = 101.04 > 1$ and $\left\|A^{-1}B\right\| = 0.09897 < 1$. Hence Theorem 2 implies that this problem is uniquely solvable for any $b \in \mathbb{R}^{10}$. For $b = (A - I)e$ and with the starting point

$$x_0 = [1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10]^T,$$

the obtained numerical results by Picard's method are stated in Table 1.

Iter	CPU (time)	RSD
5	0.008021	$3.9734e - 008$

Table 1.

The unique solution of this example is $x^* = e$.

Example 2. Consider the AVE in (1) where $A,\ B \in \mathbb{R}^{7\times 7}$ are given by:

$$A = \begin{bmatrix} 1 & 10 & 1 & 1 & 2 & 0 & 0 \\ 2 & 1 & 6 & 6 & 1 & 1 & 2 \\ 1 & 3 & 5 & 9 & 100 & 1500 & -5 \\ 5 & 1 & 3 & 1 & 0 & 3 & 40 \\ 3 & 3 & 8 & 2 & 2 & 0 & 2 \\ 1 & 5 & 5 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 2 & 1000 \end{bmatrix},$$

and

$$B = \begin{bmatrix} 0.5 & 0.5 & 0.05 & 0.05 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0.25 & 0.5 & 0.25 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0 & 0 & 0.05 & 0 \\ 0.5 & 0.05 & 0 & 0.05 & 0 & 0 & 0 \end{bmatrix}.$$

Applying Theorem 1, we have, $\sigma_{\min}(A) = 1.5029 > \sigma_{\max}(B) = 1.4653$ and $\left\|A^{-1}B\right\| = 0.373 < 1$, then this problem is uniquely solvable for any b.
For

$$b = [-16.2,\ 23,\ 3206,\ 79,\ 13,\ -1.1,\ 2004.8]^T.$$

The starting point is taken as:

$$x_0 = [1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7]^T.$$

Then the obtained numerical results are summarized in Table 2.

Iter	CPU (time)	RSD
6	0.023814	$7.1883e - 007.$

Table 2.

The exact unique solution of this problem is given by

$$x^* = [-2,\, -2,\, 2,\, 2,\, 2,\, 2,\, 2]^T.$$

Example 3. Consider the standard LCP where $M \in \mathbb{R}^{4\times 4}$ is given as

$$M = \begin{bmatrix} 0.4974 & -0.0105 & -0.0630 & -0.001 \\ -0.0839 & 0.6642 & -0.0147 & -0.00336 \\ -0.0105 & -0.042 & 0.7482 & -0.0042 \\ -0.001 & -0.0042 & -0.0252 & 0.7996 \end{bmatrix}.$$

By a simple calculation using Theorem 5, we get $\sigma_{\min}(I+M) = 1.4786 > \sigma_{\max}(I-M) = 0.5239$ and $\|(I+M)^{-1}(I-M)\| = 0.35413 < 1$ then the associated AVE has a unique solution for any $b = -q$ and consequently the LCP problem has a unique solution for every q. For example if $b = -(I+M)^{-1}q$, where $q = [-1.5,\, -2,\, -3.5,\, -4.5]^T$ and with the starting point

$$z_0 = [1,\, 2,\, 3,\, 4]^T,$$

the obtained numerical results are stated in Table 3.

Iter	CPU (time)	RSD
13	0.005759	$3.9225e-007$.

Table 3.

The exact unique solution for the AVE (7) is given by

$$z^* = [1.8664,\, 1.8110,\, 2.4831,\, 2.9040]^T.$$

Hence the unique solution of the LCP is $x^* = z^* + |z^*|$. But since z^* is positive vector i.e., $z_i^* > 0$ for all i, then $|z^*| = z^*$ and so the unique solution of the LCP is $x^* = 2z^*$.

Example 4. Consider the standard LCP where $M \in \mathbb{R}^{n\times n}$ is given by:

$$M = \begin{bmatrix} 0.6 & -0.01 & 0 & \cdots & 0 & 0 \\ -0.01 & 0.6 & -0.01 & \cdots & 0 & 0 \\ 0 & -0.01 & 0.6 & \cdots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & -0.01 & 0 \\ 0 & 0 & 0 & \cdots & 0.6 & -0.01 \\ 0 & 0 & \cdots & 0 & -0.01 & 0.6 \end{bmatrix}.$$

By a simple calculation using Theorem 5, we get for any size of n of the matrix M, $\sigma_{\min}(I+M) = 1.5808 > \sigma_{\max}(I-M) = 0.4192$, and $\|(I+M)^{-1}(I-M)\| = 0.2652 < 1$, then the LCP has a unique solution for any $q \in \mathbb{R}^n$. For $b = -(I+M)^{-1}q$ where $q = -e$, and the starting point $z_0 = [1,\, 2,\, 3, \cdots, n]^T$, the obtained numerical results with different size of n, are summarized in Table 4.

size n	Iter	CPU(s)	RSD
100	15	0.039046	$5.8061e-007$
1000	17	4.912430	$4.1563e-007$
2000	17	35.421034	$8.3189e-007$
3000	18	119.553462	$3.3178e-007$

Table 4.

Therefore the unique solution of the LCP for any size of n, is deduced from the formula $x^* = |z^*| + z^*$. Also since $z^* > 0$, then $x^* = 2z^*$ where

$$z^* = [0.8477,\ 0.8618,\ 0.8621,\ \cdots,0.8621,\ 0.8618,\ 0.8477]^T.$$

Example 5. Consider the following horizontal LCP where $M, N \in \mathbb{R}^{n\times n}$ are given by

$$M = \begin{bmatrix} 4 & 2 & 2 & \cdots & 2 & 2 \\ 2 & 4 & 2 & \cdots & 2 & 2 \\ 2 & 2 & 4 & \cdots & 2 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 2 & 2 \\ 2 & 2 & 2 & \cdots & 4 & 2 \\ 2 & 2 & \cdots & 2 & 2 & 4 \end{bmatrix}$$

and

$$N = \begin{bmatrix} 5 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 5 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 5 & \cdots & 1 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 5 & 1 \\ 1 & 1 & \cdots & 1 & 1 & 5 \end{bmatrix}.$$

By simple calculation using Theorem 6, and for different size of n, we have $\left\|(N+M)^{-1}(N-M)\right\| = 0.3333 < 1$, then this problem has a unique solution for any b. For $b = -(I+M)^{-1}q$ with $q = -Me$, and with the starting point

$$z_0 = [1, 2, 3, \cdots, n]^T,$$

the obtained numerical results for different size of n, are summarized in Table 5.

size(n)	Iter	CPU(s)	RSD
100	17	0.042596	$5.9651e-007$
1000	19	5.722172	$8.7750e-007$
2000	20	40.834046	$3.7460e-007$
3000	20	135.925389	$8.9758e-007$

Table 5.

Also since z^* is positive, the unique solution of the HLCP is $x^* = 2z^* = e$ where

$$z^* = [0.5,\ 0.5, \cdots,\ 0.5]^T.$$

6 Conclusion

In this paper, we have presented some weaker sufficient conditions that guarantee the unique solvability of the AVE (1). Our proofs are simple and elegant with the advantage that we do not use the theory of linear complementarity to prove unique solvability of the AVE (1). Across an example of AVE, we have showed the reliability of our weaker sufficient conditions to detect unique solvability of AVE (1). These obtained results are also extended to detecting unique solvability of standard and horizontal LCP. Numerically, the proposed Picard's iterative method is efficient for providing an approximated solution of some uniquely solvable AVE including standard and horizontal LCP problems.

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