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CERTAIN SUBCLASS OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH THE CHEBYSHEV POLYNOMIALS BASED ON q-DERIVATIVE AND SYMMETRIC q-DERIVATIVE

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Abstract

In this paper, we introduce new subclass $\widetilde{\mathcal{B}}^{q}_{\sigma_{\mathbb{B}}}(\lambda, t)$ of bi-univalent functions by applying the Chebyshev polynomials. In the following, we obtain bounds for the initial coefficients and the Fekete-Szegö inequalities for functions in this subclass. The results presented in this paper generalize the recent work of Altinkaya and Yalçın.

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1 Introduction

Denote $\mathcal{A}$ be the class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, satisfying the conditions

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1.$$

Then each function $f \in \mathcal{A}$ has the Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

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Further, we let $\mathcal{S}$ to denote the class of functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$ (see details [3, 4, 8, 12]). Also, function f is said to be bi-univalent, If both f and f^{-1} belong to $\mathcal{S}$. We denote $\sigma_{\mathcal{B}}$ for the class of bi-univalent functions in $\mathbb{U}$. Also every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \qquad \left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right).$$

Also, the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \tag{2}$$

We say that f is subordinate to F in $\mathbb{U}$, written as $f \prec F$ $(z \in \mathbb{U})$, if and only if $f(z) = F(w(z))$ for some Schwarz function w(z) such that:

$$w(0) = 0 \text{ and } |w(z)| < 1 \ (z \in \mathbb{U}).$$

If F is univalent in $\mathbb{U}$, then the subordination $f \prec F$ is equivalent to $f(0) = F(0)$ and $f(\mathbb{U}) \subset F(\mathbb{U})$.

The important roll of Chebyshev polynomial in numerical analysis is increased in both theoretical and practical points of view. There are four kinds of Chebyshev polynomials. Many researchers dealing with orthogonal polynomials of Chebyshev.

Doha [7] expressed a brief history of Chebyshev polynomials of the first kind $T_n(t)$, the second kind $U_n(t)$ and their applications (for details see [9, 11]).

The Chebyshev polynomials of the first and second kinds are well known. In the case of a real variable x on $(-1, 1)$, they are defined by

$$T_n(x) = cosn\theta \text{ and } U_n(x) = \frac{sin(n+1)\theta}{sin\theta},$$

where n denotes the polynomial degree and $x = cos\theta$.

Lemma 1. *[8] If $p(z) \in P$, then $|c_k| \leq 2$ for each k, where P is the family of all functions $p(z)$ analytic in $\mathbb{U}$ for which*

$$\Re\ p(z) > 0, \quad p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots \ for\ z \in \mathbb{U}.$$

Definition 1. *[10] For a function $f \in \mathcal{A}$ given by (1) and $0 < q < 1$, the q-derivative of function f is defined by*

$$D_qf(z) = \begin{cases} \frac{f(qz)-f(z)}{(q-1)z} \ ; & z \neq 0 \\ f'(0) \ ; & z = 0. \end{cases}$$

According to the definition, we conclude that

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \tag{3}$$

where

$$[n]_q = \frac{1-q^n}{1-q}. \tag{4}$$

If $q \to 1^-$, *then* $[n]_q \to n$. *We note that* $\lim_{q\to 1^-} D_q f(z) = f'(z)$, *for* $f \in \mathcal{A}$.

Definition 2. *[2] For a function* $f \in \mathcal{A}$ *given by* (1) *and* $0 < q < 1$, *the symmetric* q*-derivative of function* f *is defined by*

$$\widetilde{D}_q f(z) = \begin{cases} \frac{f(qz)-f(q^{-1}z)}{(q-q^{-1})z} & ; \quad z \neq 0 \\ f'(0) & ; \quad z = 0. \end{cases}$$

According to the definition, we conclude that $\widetilde{D}_q z^n = \widetilde{[n]}_q z^{n-1}$, *and a power series of* $\widetilde{D}_q f(z)$ *is*

$$\widetilde{D}_q f(z) = 1 + \sum_{n=2}^{\infty} \widetilde{[n]}_q a_n z^{n-1}, \tag{5}$$

where

$$\widetilde{[n]}_q = \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{6}$$

If $q \to 1^-$, *then* $\widetilde{[n]}_q \to n$. *We note that* $\lim_{q\to 1^-} \widetilde{D}_q f(z) = f'(z)$, *for* $f \in \mathcal{A}$.

2 Coefficient bounds for the function subclass $\widetilde{\mathcal{B}}^q_{\sigma_{\mathbb{B}}}(\lambda, t)$

In this part, we present the subclass $\widetilde{\mathcal{B}}^q_{\sigma_{\mathbb{B}}}(\lambda, t)$ of bi-univalent functions by using the Chebyshev polynomial expansions. In the following, we obtain the coefficients estimates for functions in this subclass. Also the Fekete-Szegö problem in this subclass is found.

Definition 3. *Let* λ, q *and* t *be real numbers such that* $0 \leq \lambda \leq 1$, $0 < q < 1$ *and* $t \in (\frac{1}{2}, 1]$. *The bi-univalent function* f *is said to be in the class* $\widetilde{\mathcal{B}}^q_{\sigma_{\mathbb{B}}}(\lambda, t)$, *if the following conditions are satisfied:*

$$\left(\lambda D_q f(z) + (1-\lambda)\widetilde{D}_q f(z)\right) \prec H(z,t) := \frac{1}{1 - 2tz + z^2} \ (z \in \mathbb{U})$$

and

$$\left(\lambda D_q g(w) + (1-\lambda)\widetilde{D}_q g(w)\right) \prec H(w,t) := \frac{1}{1 - 2tw + w^2} \ (w \in \mathbb{U}),$$

where the function $g = f^{-1}$ *is defined by* (2).

If we put $t = cos\theta$, where $\theta \in (-\frac{\pi}{3}, \frac{\pi}{3})$, then

$$H(z,t) = \frac{1}{1-2cos\theta z + z^2} = 1 + \sum_{n=1}^{\infty} \frac{sin(n+1)\theta}{sin\theta} z^n \quad (z \in \mathbb{U}).$$

Thus

$$H(z,t) = 1 + 2cos\theta z + (3cos^2\theta - sin^2\theta)z^2 + \cdots \quad (z \in \mathbb{U}).$$

So we can write

$$H(z,t) = 1 + U_1(t)z + U_2(t)z^2 + \cdots \quad (z \in \mathbb{U},\ t \in (-1,1))$$

where $U_{n-1}(t) = \frac{sin(n\ arccos\ t)}{\sqrt{1-t^2}}$ $(n \in \mathbb{N})$ are the Chebyshev polynomials of the second kind and we get

$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t)$$

and

$$U_1(t) = 2t, \quad U_2(t) = 4t^2 - 1, \quad U_3(t) = 8t^3 - 4t, \quad U_4(t) = 16t^4 - 12t^2 + 1, \cdots. \quad (7)$$

The Chebyshev polynomials of the first kind can be obtained from the generating functions

$$\sum_{n=0}^{\infty} T_n(t)z^n = \frac{1-tz}{1-2tz+z^2} \quad (z \in \mathbb{U}).$$

However, the Chebyshev polynomials of the first kind $T_n(t)$ and the second kind $U_n(t)$ are linked by the following connections

$$\frac{dT_n(t)}{dt} = nU_{n-1}(t) \qquad and \qquad T_n(t) = U_n(t) - tU_{n-1}(t).$$

Remark 1. *By putting $\lambda = 0$, then the class $\widetilde{\mathcal{B}}^q_{\sigma_{\mathbb{B}}}(\lambda,t)$ reduces to the class $\widetilde{\mathcal{H}}^q_{\Sigma}(t)$ that was introduced and studied by Altinkaya et al. [1].*

By taking $q \to 1^-$ and $\lambda = 0$, then the class $\widetilde{\mathcal{B}}^q_{\sigma_{\mathbb{B}}}(\lambda,t)$ reduces to the class $\mathcal{H}_{\Sigma}(t)$ that was introduced and studied by Altinkaya et al. [1].

By taking $q \to 1^-$ and $\lambda = 1$, then the class $\widetilde{\mathcal{B}}^q_{\sigma_{\mathbb{B}}}(\lambda,t)$ reduces to the class $\mathcal{H}_{\Sigma}(t)$ that was introduced and studied by Altinkaya et al. [1].

Theorem 1. *Let function f given by* (1) *be in the class $\widetilde{\mathcal{B}}^q_{\sigma_{\mathbb{B}}}(\lambda,t)$. Then*

$$|a_2| \leq \min\left\{ \frac{2t}{\lambda[2]_q + (1-\lambda)\widetilde{[2]}_q}, \right.$$

$$\left.\frac{2t\sqrt{2t}}{\sqrt{\left|4\left(\lambda[3]_q+(1-\lambda)\widetilde{[3]}_q-(\lambda[2]_q+(1-\lambda)\widetilde{[2]}_q)^2\right)t^2+(2t+1)(\lambda[2]_q+(1-\lambda)\widetilde{[2]}_q)^2\right|}}\right\},$$

$$|a_3| \leq \frac{4t^2}{(\lambda[2]_q + (1-\lambda)\widetilde{[2]}_q)^2} + \frac{2t}{\lambda[3]_q + (1-\lambda)\widetilde{[3]}_q}$$

and

$$|a_4| \leq \begin{cases} \frac{8t-8t^3}{\lambda[4]_q+(1-\lambda)\widetilde{[4]}_q} + \frac{10t^2}{(\lambda[2]_q+(1-\lambda)\widetilde{[2]}_q)(\lambda[3]_q+(1-\lambda)\widetilde{[3]}_q)} & ; \quad \frac{1}{2} \leq t \leq \frac{1+\sqrt{5}}{4} \\ \frac{-4+16t^2-8t^3}{\lambda[4]_q+(1-\lambda)\widetilde{[4]}_q} + \frac{10t^2}{(\lambda[2]_q+(1-\lambda)\widetilde{[2]}_q)(\lambda[3]_q+(1-\lambda)\widetilde{[3]}_q)} & ; \quad \frac{1+\sqrt{5}}{4} \leq t < 1. \end{cases}$$

Proof. Suppose that the function $f \in \sigma_{\mathbb{B}}$ given by (1) be in the class $\widetilde{\mathcal{B}}^q_{\sigma_{\mathbb{B}}}(\lambda, t)$. Then there exist two functions u and v, analytic in $\mathbb{U}$, with $u(0) = v(0) = 0$, $|u(z)| < 1$ and $|v(w)| < 1$, such that:

$$\begin{aligned} \lambda D_q f(z) + (1-\lambda)\widetilde{D}_q f(z) &= H\left(u(z), t\right) \\ &= 1 + U_1(t)u(z) + U_2(t)u(z)^2 + \cdots \end{aligned} \tag{8}$$

and

$$\begin{aligned} \lambda D_q g(w) + (1-\lambda)\widetilde{D}_q g(w) &= H\left(v(w), t\right) \\ &= 1 + U_1(t)v(w) + U_2(t)v(w)^2 + \cdots . \end{aligned} \tag{9}$$

Next, define the functions h and k by

$$h(z) = \frac{1+u(z)}{1-u(z)} = 1 + h_1 z + h_2 z^2 + h_3 z^3 + \cdots \tag{10}$$

and

$$k(w) = \frac{1+v(w)}{1-v(w)} = 1 + k_1 w + k_2 w^2 + k_3 w^3 + \cdots . \tag{11}$$

Since u and v are Schwarz functions, h and k are analytic functions in $\mathbb{U}$, with $h(0) = k(0) = 1$ and which have positive real part in $\mathbb{U}$. From the equations (10) and (11), we get

$$u(z) = \frac{h(z)-1}{h(z)+1} = \frac{h_1}{2}z + \frac{1}{2}(h_2 - \frac{h_1^2}{2})z^2 + (h_3 - h_1 h_2 + \frac{h_1^3}{4})z^3 + \cdots \tag{12}$$

and

$$v(w) = \frac{k(w)-1}{k(w)+1} = \frac{k_1}{2}w + \frac{1}{2}(k_2 - \frac{k_1^2}{2})w^2 + (k_3 - k_1 k_2 + \frac{k_1^3}{4})w^3 + \cdots . \tag{13}$$

From (8), (9), (12) and (13), we have

$$\begin{aligned}\lambda D_q f(z)+(1-\lambda)\widetilde{D}_q f(z) &= 1+\frac{U_1(t)}{2}h_1 z+\left(\frac{U_1(t)}{2}(h_2-\frac{h_1^2}{2})+\frac{U_2(t)}{4}h_1^2\right)z^2\\ &\quad+\Big(\frac{U_1(t)}{2}(h_3-h_1h_2+\frac{h_1^3}{4})+\frac{U_2(t)}{2}h_1(h_2-\frac{h_1^2}{2})\\ &\quad+\frac{U_3(t)}{8}h_1^3\Big)z^3+\cdots \qquad (14)\end{aligned}$$

and

$$\begin{aligned}\lambda D_q g(w)+(1-\lambda)\widetilde{D}_q g(w) &= 1+\frac{U_1(t)}{2}k_1 w+\left(\frac{U_1(t)}{2}(k_2-\frac{k_1^2}{2})+\frac{U_2(t)}{4}k_1^2\right)w^2\\ &\quad+\Big(\frac{U_1(t)}{2}(k_3-k_1k_2+\frac{k_1^3}{4})+\frac{U_2(t)}{2}k_1(k_2-\frac{k_1^2}{2})\\ &\quad+\frac{U_3(t)}{8}k_1^3\Big)w^3+\cdots. \qquad (15)\end{aligned}$$

From (14) and (15), we get

$$(\lambda[2]_q+(1-\lambda)\widetilde{[2]}_q)a_2 = \frac{U_1(t)}{2}h_1, \qquad (16)$$

$$(\lambda[3]_q+(1-\lambda)\widetilde{[3]}_q)a_3 = \frac{U_1(t)}{2}(h_2-\frac{h_1^2}{2})+\frac{U_2(t)}{4}h_1^2, \qquad (17)$$

$$\begin{aligned}(\lambda[4]_q+(1-\lambda)\widetilde{[4]}_q)a_4 &= \frac{U_1(t)}{2}(h_3-h_1h_2+\frac{h_1^3}{4})\\ &\quad+\frac{U_2(t)}{2}h_1(h_2-\frac{h_1^2}{2})+\frac{U_3(t)}{8}h_1^3, \qquad (18)\end{aligned}$$

$$-(\lambda[2]_q+(1-\lambda)\widetilde{[2]}_q)a_2 = \frac{U_1(t)}{2}k_1, \qquad (19)$$

$$(\lambda[3]_q+(1-\lambda)\widetilde{[3]}_q)(2a_2^2-a_3) = \frac{U_1(t)}{2}(k_2-\frac{k_1^2}{2})+\frac{U_2(t)}{4}k_1^2 \qquad (20)$$

and

$$\begin{aligned}-(\lambda[4]_q+(1-\lambda)\widetilde{[4]}_q)(5a_2^3-5a_2a_3+a_4) &= \frac{U_1(t)}{2}(k_3-k_1k_2+\frac{k_1^3}{4})\\ &\quad+\frac{U_2(t)}{2}k_1(k_2-\frac{k_1^2}{2})+\frac{U_3(t)}{8}k_1^3. \qquad (21)\end{aligned}$$

From (16) and (19), we have

$$h_1=-k_1 \qquad (22)$$

and

$$a_2=\frac{U_1(t)h_1}{2(\lambda[2]_q+(1-\lambda)\widetilde{[2]}_q)} \qquad (23)$$

Now, from (17) and (20), we obtain

$$2(\lambda[3]_q + (1-\lambda)\widetilde{[3]}_q)a_2^2 = \frac{U_1(t)}{2}(h_2+k_2) + \frac{U_2(t)-U_1(t)}{4}(h_1^2+k_1^2). \tag{24}$$

By using (23) in (24), we obtain

$$2\left[(\lambda[3]_q + (1-\lambda)\widetilde{[3]}_q)U_1^2(t) + (U_1(t)-U_2(t))(\lambda[2]_q + (1-\lambda)\widetilde{[2]}_q)^2\right] a_2^2 = \frac{U_1^3(t)}{2}(h_2+k_2). \tag{25}$$

So, applying Lemma 1 and substituting the values of $U_1(t),\ U_2(t)$ from (7) into (23) and (25), we immediately have

$$|a_2| \leq \frac{2t}{\lambda[2]_q + (1-\lambda)\widetilde{[2]}_q}$$

and

$$|a_2|^2 \leq \frac{8t^3}{\left|4\left(\lambda[3]_q + (1-\lambda)\widetilde{[3]}_q - (\lambda[2]_q + (1-\lambda)\widetilde{[2]}_q)^2\right)t^2 + (2t+1)(\lambda[2]_q + (1-\lambda)\widetilde{[2]}_q)^2\right|}.$$

Next, in order to obtain the bound on $|a_3|$, by subtracting (20) from (17), we have

$$a_3 = a_2^2 + \frac{U_1(t)}{4(\lambda[3]_q + (1-\lambda)\widetilde{[3]}_q)}(h_2-k_2). \tag{26}$$

By using (23) in (26), we have

$$a_3 = \frac{h_1^2U_1^2(t)}{4(\lambda[2]_q + (1-\lambda)\widetilde{[2]}_q)^2} + \frac{(h_2-k_2)U_1(t)}{4(\lambda[3]_q + (1-\lambda)\widetilde{[3]}_q)}. \tag{27}$$

So, by using Lemma 1 again, we obtain

$$|a_3| \leq \frac{4t^2}{(\lambda[2]_q + (1-\lambda)\widetilde{[2]}_q)^2} + \frac{2t}{\lambda[3]_q + (1-\lambda)\widetilde{[3]}_q}.$$

Finally, to determine the bounds on a_4, Subtracte (21) from (18) with $h_1 = -k_1$, we obtain

$$\begin{aligned} a_4 = &\\ = &\frac{2U_1(t)(h_3-k_3) + 2(U_2(t)-U_1(t))(h_2+k_2)h_1 + h_1^3(U_1(t)-2U_2(t)+U_3(t))}{8(\lambda[4]_q + (1-\lambda)\widetilde{[4]}_q)} \\ &+ \frac{5h_1(h_2-k_2)U_1^2(t)}{16(\lambda[2]_q + (1-\lambda)\widetilde{[2]}_q)(\lambda[3]_q + (1-\lambda)\widetilde{[3]}_q)}. \end{aligned} \tag{28}$$

By substituting the values of $U_1(t),\ U_2(t)$ from (7) into (28), we have

$$\begin{aligned} a_4 &= \frac{2t(h_3-k_3)+(4t^2-2t-1)h_1(h_2+k_2)+h_1^3(4t^3-4t^2-t+1)}{4(\lambda[4]_q+(1-\lambda)\widetilde{[4]}_q)} \\ &\quad + \frac{5h_1(h_2-k_2)t^2}{4(\lambda[2]_q+(1-\lambda)\widetilde{[2]}_q)(\lambda[3]_q+(1-\lambda)\widetilde{[3]}_q)}. \end{aligned} \tag{29}$$

So, by using Lemma 1, we get

$$\begin{aligned} |a_4| &\leq \frac{2t+2|4t^2-2t-1|+2|4t^3-4t^2-t+1|}{\lambda[4]_q+(1-\lambda)\widetilde{[4]}_q} \\ &\quad + \frac{10t^2}{(\lambda[2]_q+(1-\lambda)\widetilde{[2]}_q)(\lambda[3]_q+(1-\lambda)\widetilde{[3]}_q)}. \end{aligned}$$

□

Now, we consider functional $|a_3-\rho a_2^2|$ for real ρ.

Theorem 2. *Let f given by (1) be in the class $\widetilde{\mathcal{B}}^q_{\sigma_{\mathbb{B}}}(\lambda,t)$ and $\rho\in\mathbb{R}$. Then*

$$|a_3-\rho a_2^2| \leq \begin{cases} \frac{2t}{\lambda[3]_q+(1-\lambda)\widetilde{[3]}_q} & ;\ |h(\rho)| \leq \frac{1}{4(\lambda[3]_q+(1-\lambda)\widetilde{[3]}_q)t^2} \\ 8t|h(\rho)| & ;\ |h(\rho)| \geq \frac{1}{4(\lambda[3]_q+(1-\lambda)\widetilde{[3]}_q)t^2}, \end{cases}$$

where
$h(\rho) =$

$$= \frac{t^2(1-\rho)}{4t^2\left[\lambda[3]_q+(1-\lambda)\widetilde{[3]}_q-(\lambda[2]_q+(1-\lambda)\widetilde{[2]}_q)^2\right]+(2t+1)(\lambda[2]_q+(1-\lambda)\widetilde{[2]}_q)^2}.$$

Proof. From (23) and (24) for some $\rho\in\mathbb{R}$, we get

$$\begin{aligned} a_3-\rho a_2^2 &= \frac{U_1^3(t)(h_2+k_2)(1-\rho)}{4\left[(\lambda[3]_q+(1-\lambda)\widetilde{[3]}_q)U_1^2(t)+(U_1(t)-U_2(t))(\lambda[2]_q+(1-\lambda)\widetilde{[2]}_q)^2\right]} \\ &\quad + \frac{U_1(t)(h_2-k_2)}{4(\lambda[3]_q+(1-\lambda)\widetilde{[3]}_q)}. \end{aligned}$$

Hence

$$\begin{aligned} a_3-\rho a_2^2 &= U_1(t)\Bigg[\left(h(\rho)+\frac{1}{4(\lambda[3]_q+(1-\lambda)\widetilde{[3]}_q)}\right)h_2 \\ &\quad + \left(h(\rho)-\frac{1}{4(\lambda[3]_q+(1-\lambda)\widetilde{[3]}_q)}\right)k_2\Bigg]. \end{aligned}$$

where

$$h(\rho) = \frac{U_1^2(t)(1-\rho)}{4\left[(\lambda[3]_q + (1-\lambda)\widetilde{[3]}_q)U_1^2(t) + (U_1(t) - U_2(t))(\lambda[2]_q + (1-\lambda)\widetilde{[2]}_q)^2\right]}$$

$$= \frac{t^2(1-\rho)}{4t^2\left[\lambda[3]_q + (1-\lambda)\widetilde{[3]}_q - (\lambda[2]_q + (1-\lambda)\widetilde{[2]}_q)^2\right] + (2t+1)(\lambda[2]_q + (1-\lambda)\widetilde{[2]}_q)^2}.$$

So, by using Lemma 1, we obtain

$$\begin{aligned}|a_3 - \rho a_2^2| \leq 2|U_1(t)|\Bigg\{&\left|h(\rho) + \frac{1}{4(\lambda[3]_q + (1-\lambda)\widetilde{[3]}_q)}\right| \\ &+ \left|h(\rho) - \frac{1}{4(\lambda[3]_q + (1-\lambda)\widetilde{[3]}_q)}\right|\Bigg\}.\end{aligned}$$

Therefore, we derive that

$$|a_3 - \rho a_2^2| \leq \begin{cases} \frac{2t}{\lambda[3]_q + (1-\lambda)\widetilde{[3]}_q} & ; \quad |h(\rho)| \leq \frac{1}{4(\lambda[3]_q + (1-\lambda)\widetilde{[3]}_q)} \\ 8t|h(\rho)| & ; \quad |h(\rho)| \geq \frac{1}{4(\lambda[3]_q + (1-\lambda)\widetilde{[3]}_q)}. \end{cases}$$

□

3 Corollaries and Consequences

By putting $\lambda = 0$ in Theorem 1, we conclude the following corollary.

Corollary 1. *Let the function f given by (1) be in the class $\widetilde{\mathcal{H}}_\Sigma^q(t)$. Then*

$$|a_2| \leq \min\left\{\frac{2t\sqrt{2t}}{\sqrt{\left|4\left(\widetilde{[3]}_q - \widetilde{[2]}_q^2\right)t^2 + (2t+1)\widetilde{[2]}_q^2\right|}}, \frac{2t}{\widetilde{[2]}_q}|\right\},$$

$$|a_3| \leq \frac{4t^2}{\widetilde{[2]}_q^2} + \frac{2t}{\widetilde{[3]}_q}$$

and

$$|a_4| \leq \begin{cases} \frac{8t-8t^3}{\widetilde{[4]}_q} + \frac{10t^2}{\widetilde{[2]}_q\widetilde{[3]}_q} & ; \quad \frac{1}{2} \leq t \leq \frac{1+\sqrt{5}}{4} \\ \frac{-4+16t^2-8t^3}{\widetilde{[4]}_q} + \frac{10t^2}{\widetilde{[2]}_q\widetilde{[3]}_q} & ; \quad \frac{1+\sqrt{5}}{4} \leq t < 1. \end{cases}$$

By putting $\lambda = 0$ in Theorem 2, we conclude the following corollary.

Corollary 2. *Let the function f given by (1) be in the class $\widetilde{\mathcal{H}}_{\Sigma}^{q}(t)$ and $\rho \in \mathbb{R}$. Then*

$$|a_3 - \rho a_2^2| \leq \begin{cases} \frac{2t}{\widetilde{[3]}_q} & ; \quad |1-\rho| \leq \frac{\left|4\left(\widetilde{[3]}_q - \widetilde{[2]}_q^2\right)t^2 + (2t+1)\widetilde{[2]}_q^2\right|}{4\widetilde{[3]}_q t^2} \\ \frac{8|1-\rho|t^3}{\left|4\left(\widetilde{[3]}_q - \widetilde{[2]}_q^2\right)t^2 + (2t+1)\widetilde{[2]}_q^2\right|}; & \quad |1-\rho| \geq \frac{\left|4\left(\widetilde{[3]}_q - \widetilde{[2]}_q^2\right)t^2 + (2t+1)\widetilde{[2]}_q^2\right|}{4\widetilde{[3]}_q t^2}. \end{cases}$$

By taking $q \to 1^-$ in Corollary 1, we conclude the following corollary.

Corollary 3. *Let the function f given by (1) be in the class $\mathcal{H}_{\Sigma}(t)$. Then*

$$|a_2| \leq \min\left\{ \frac{t\sqrt{2t}}{\sqrt{1+2t-t^2}}, t \right\}, \qquad |a_3| \leq t^2 + \frac{2t}{3}$$

and

$$|a_4| \leq \begin{cases} 2t - 2t^3 + \frac{5t^2}{3} & ; \quad \frac{1}{2} \leq t \leq \frac{1+\sqrt{5}}{4} \\ -1 + 4t^2 - 2t^3 + \frac{5t^2}{3} & ; \quad \frac{1+\sqrt{5}}{4} \leq t < 1. \end{cases}$$

By taking $q \to 1^-$ in Corollary 3, we conclude the following corollary.

Corollary 4. *Let the function f given by (1) be in the class $\mathcal{H}_{\Sigma}(t)$ and $\rho \in \mathbb{R}$. Then*

$$|a_3 - \rho a_2^2| \leq \begin{cases} \frac{2t}{3} & ; \quad |1-\rho| \leq \frac{1+2t-t^2}{3t^2} \\ \frac{2|1-\rho|t^3}{1+2t-t^2} & ; \quad |1-\rho| \geq \frac{1+2t-t^2}{3t^2}. \end{cases}$$

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