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COEFFICIENT ESTIMATES FOR CERTAIN SUBCLASSES OF STARLIKE FUNCTIONS OF COMPLEX ORDER ASSOCIATED WITH A HYPERBOLIC DOMAIN

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Abstract

In this paper, we obtain the coefficient inequalities for functions in certain subclasses of Janowski starlike functions of complex order which are related starlike functions associated with a hyperbolic domain. Our results extend the study of various subclasses of analytic functions. Several applications of our results are also mentioned.

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1 Introduction

Let $\mathcal{A}$ denote the class of functions analytic in $\mathbb{U} = \{z : |z| < 1\}$ and having a Taylor series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} c_n z^n \quad c_n \geq 0. \tag{1}$$

Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ of analytic and univalent functions in $\mathcal{U}$. Also, let $\mathcal{S}^*(\eta)$ and $\mathcal{C}(\eta)$ denote the familiar subclasses of $\mathcal{A}$ consisting of functions which are respectively starlike of order η and convex of order η in $\mathbb{U}$. The class $\mathcal{P}$ denotes the class of functions of the form $p(z) = 1+p_1z+p_2z^2+p_3z^3+\cdots$ that are analytic

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in $\mathbb{U}$ and such that $Re\,(p(z)) > 0$ for all z in $\mathbb{U}$. For detailed study of various subclasses of univalent function theory, we refer to [5].

The functions $p_\alpha(z)$ plays the role of extremal functions related to the hyperbolic domains (see [14, 21]) and is given by

$$p_\alpha(z) = (1+2\alpha)\sqrt{\frac{1+bz}{1-bz}} - 2\alpha, \tag{2}$$

where

$$b = b(\alpha) = \frac{1+4\alpha-4\alpha^2}{(1+2\alpha)^2}, \quad \alpha > 0$$

the branch of the square root $\sqrt{w}$ being chosen such that $Re\sqrt{w} \geq 0$. Clearly, $p_\alpha(z) \in \mathcal{P}$ is analytic with the expansion of the form

$$p_\alpha(z) = 1 + L_1 z + L_2 z^2 + \cdots, \qquad (L_j = p_j(\alpha),\ j = 1,\ 2,\ 3, \ldots), \tag{3}$$

where

$$L_1 = \frac{(1+4\alpha)}{1+2\alpha} \qquad \text{and} \qquad L_2 = \frac{(1+4\alpha)(1+4\alpha+8\alpha^2)}{2(1+2\alpha)^3}.$$

In [21], Stankiewicz and Wiśniowska defined $SH(\alpha)$ as the class of functions $f \in \mathcal{A}$ satisfying the condition

$$\left|\frac{zf'(z)}{f(z)} - 2\alpha\left(\sqrt{2}-1\right)\right| < \sqrt{2}Re\left(\frac{zf'(z)}{f(z)}\right) + 2\alpha\left(\sqrt{2}-1\right),$$

for some $\alpha > 0$. Note that $f \in SH(\alpha)$ if and only if $\frac{zf'(z)}{f(z)}$ lies in the hyperbolic domain

$$\Omega(\alpha) = \left\{w = u + iv : v^2 < 4\alpha u + u^2,\ u > 0\right\}$$

which is included in the right half-plane and is symmetric to the real axis with a vertex at the origin. Equivalently, a function $f \in SH(\alpha)$ if and only if

$$\frac{zf'(z)}{f(z)} \prec p_\alpha(z),$$

where $\prec$ denotes the subordination. The function $p_\alpha(z)$ maps the unit disk conformally onto the domain $\Omega(\alpha)$ such that $p_\alpha(0) = 1$ and $p'_\alpha(0) > 0$. The class $SH(\alpha)$ was motivated by the study of uniformly convex and uniformly starlike functions (see [6, 9, 10, 17]).

Now we briefly recall the q-calculus and the notations which are required for our study. Quantum calculus (*q-calculus* and *h-calculus*) is common classical calculus without the notion of limits. Here, h represents Planckís constant, while q represents quantum. Due to its application in a variety of branches such as physics, mathematics, the area of *q-calculus* has gained great importance for researchers. The first study on *q-calculus* was systematically established by Jackson [8] as q-derivative is merely a ratio which is given by

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}.$$

Note that $\lim_{q\to 1^-} D_q f(z) = f'(z)$. Notations and symbols play a very important role in the study of q-calculus. Throughout this paper, we let

$$[n]_q = \sum_{k=1}^{n} q^{k-1}, \quad [0]_q = 0, \quad (q \in \mathbb{C})$$

and the q-shifted factorial by

$$(a;q)_n = \begin{cases} 1, & n = 0 \\ (1-a)(1-aq)\dots\left(1-aq^{n-1}\right), & n = 1, 2, \dots. \end{cases}$$

In [7], Ismail et. al. introduced the class $\mathcal{S}_q^*$ to be the class of functions which satisfy the condition

$$\left| \frac{zD_q f(z)}{f(z)} - \frac{1}{1-q} \right| \le \frac{1}{1-q}, \qquad (f \in \mathcal{S}).$$

Equivalently, a function $f \in \mathcal{S}_q^*$ if and only if the following subordination condition (see [13, 20])

$$\frac{zD_q f(z)}{f(z)} \prec \frac{1+z}{1-qz}$$

holds. It can easily be seen that if $q \to 1^-$, $\mathcal{S}_q^*$ reduces to the well-known class of starlike functions.

Recently in [15], the authors defined the following q-differential operator $\mathfrak{J}_\lambda^m(a_1, b_1;\, q,\, z)f : \mathbb{U} \to \mathbb{U}$ given by

$$\mathfrak{J}_\lambda^m(a_1, b_1;\, q,\, z)f = z + \sum_{n=2}^{\infty} [1 - \lambda + \lambda[n]_q]^m \; \Gamma_n c_n z^n, \tag{4}$$

$$(m \in N_0 = N \cup \{0\} \text{ and } \lambda \ge 0)\,,$$

where

$$\Gamma_n = \frac{(a_1;q)_{n-1}(a_2;q)_{n-1}\dots(a_r;q)_{n-1}}{(q;q)_{n-1}(b_1;q)_{n-1}\dots(b_s;q)_{n-1}}, \quad (|q| < 1)\,.$$

Remark 1. *For a detailed study and applications of the operator* $\mathfrak{J}_\lambda^m(a_1, b_1;\, q,\, z)f$*, refer to [15] and the references provided therein. We let* $\mathscr{D}^{m+1}f$ *denote the well-known Sălăgean derivative operator (see [18]), which is a special case of the operator* $\mathfrak{J}_\lambda^m(a_1, b_1;\, q,\, z)f$*.*

Throughout our present discussion, to avoid repetition, we lay down once and for all that $-1 \le B < A \le 1$, $z \in \mathbb{U}$ and Γ_n is real.

Motivated by the class $SH(\alpha)$, we define the following.

Definition 1. *For* $p_\alpha(z)$ *defined as in (2), a function* $f \in k{-}\mathcal{SH}_{q,\alpha}^{\lambda, m}(\gamma; a_1,\, b_1; A, B)$ *if and only if*

$$1 + \frac{1}{\gamma}\left(\frac{zD_q\left[\mathfrak{J}_\lambda^m(a_1, b_1;\, q,\, z)f\right]}{\mathfrak{J}_\lambda^m(a_1, b_1;\, q,\, z)f} - 1 \right) \prec \frac{(A+1)p_\alpha(z) - (A-1)}{(B+1)p_\alpha(z) - (B-1)}, \qquad (f \in \mathcal{A}) \tag{5}$$

where $\gamma \in \mathbb{C} \setminus \{0\}$*,* $k \ge 0$ *and* $0 \le \alpha < 1$*.*

Remark 2. *If we let $A = 1$, $B = -1, \gamma = 1 + 0i$, $m = 0, r = 2, s = 1, a_1 = b_1, a_2 = q$ and $q \to 1^-$, the class $k - \mathcal{SH}_{q,\alpha}^{\lambda,m}(\gamma; a_1, b_1; A, B)$ reduces to the class $SH(\alpha)$.*

Definition 2. *Let $\mathcal{S}_{q,\alpha}^{\lambda,m}(\gamma; a_1, b_1; A, B)$ denote the class of functions $f \in \mathcal{A}$ satisfying the inequality*

$$1 + \frac{1}{\gamma}\left(\frac{zD_q\left[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f\right]}{\mathcal{J}_\lambda^m(a_1, b_1; q, z)f} - 1\right) \prec \frac{(A+1)\hat{p}(z) - (A-1)}{(B+1)\hat{p}(z) - (B-1)}, \tag{6}$$

where $\hat{p}(z) = \frac{1+z}{1-qz}$, $q \in (0, 1)$, $\gamma \in \mathbb{C} \setminus \{0\}$.

Remark 3. *The study of classes $k-\mathcal{SH}_{q,\alpha}^{\lambda,m}(\gamma; a_1, b_1; A, B)$ and $\mathcal{S}_{q,\alpha}^{\lambda,m}(\gamma; a_1, b_1; A, B)$ was motivated by the Noor and Malik [12]. For geometrical interpretation and purpose to study such conic region, refer to [2, 12, 20] and the references provided therein. Several well-known classes (just to name a few) α-spiral functions, convex α-spiral functions, starlike functions of complex order and convex functions of complex order can be obtained as special cases our class $\mathcal{S}_{q,\alpha}^{\lambda,m}(\gamma; a_1, b_1; A, B)$.*

By definition of subordination, a function $f \in \mathcal{A}$ is said to be in $\mathcal{S}_{q,\alpha}^{\lambda,m}(\gamma; a_1, b_1; A, B)$ if and only if

$$1 + \frac{1}{\gamma}\left(\frac{zD_q\left[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f\right]}{\mathcal{J}_\lambda^m(a_1, b_1; q, z)f} - 1\right) = \frac{(A+1)w(z) + 2 + (A-1)qw(z)}{(B+1)w(z) + 2 + (B-1)qw(z)}, \tag{7}$$

$$(q \in (0, 1),\ z \in \mathbb{U}),$$

where $w(z)$ is analytic in $\mathbb{U}$ and $w(0) = 0$, $|w(z)| < 1$.

2 Coefficient inequalities

We need the following Lemma to prove our main result in this section.

Lemma 1. *[23] Suppose that $|(A-B)\gamma(1+q) - \{B(1+q) + (1-q)\}([n]_q - 1)| - 4([n]_q - 1) > 0$.*

$$\begin{aligned}
&(A-B)^2(1+q)^2 \mid \gamma \mid^2 \\
&+ \sum_{n=2}^{j-1}\left\{|(A-B)\gamma(1+q) - \{B(1+q) + (1-q)\}([n]_q - 1)|^2 - 4([n]_q - 1)^2\right\} \\
&\times \Omega_n^2\left\{\frac{1}{\Omega_n^2}\prod_{j=2}^{n}\frac{\mid (A-B)(1+q)\gamma - ([j-1]_q - 1)[B(1+q) + (1-q)] \mid^2}{2([j]_q - 1)^2}\right\} \\
&= \frac{1}{([j-1]_q - 1)!^2}\prod_{n=2}^{j}\frac{\mid (A-B)(1+q)\gamma - ([n-1]_q - 1)[B(1+q) + (1-q)] \mid^2}{2}
\end{aligned}$$

where $q \in (0, 1)$, $\gamma \in \mathbb{C} \setminus \{0\}$, $n \in N \setminus \{0\}$.

Theorem 1. *Let the function $f(z)$ defined by (1) be in the class $\mathcal{S}_{q,\alpha}^{\lambda,m}(\gamma; a_1, b_1; A, B)$ and let $L_n = |(A-B)\gamma(1+q) - \{B(1+q)+(1-q)\}([n]_q - 1)| - 4([n]_q - 1)$.*

(a) If $L_2 \leq 0$, then

$$|c_j| \leq \frac{(A-B)(1+q) \mid \gamma \mid}{2\left[1-\lambda+[j]_q\lambda\right]^m ([j]_q - 1)\ \Gamma_j}. \tag{8}$$

(b) If $L_n \geq 0$, then

$$\begin{aligned} \mid c_j \mid \leq & \frac{1}{\left[1-\lambda+[j]_q\lambda\right]^m \Gamma_j} \\ & \times \prod_{n=2}^{j} \frac{\mid (A-B)(1+q)\gamma - ([n-1]_q - 1)\left[B(1+q)+(1-q)\right] \mid}{2\left([n]_q - 1\right)} \end{aligned} \tag{9}$$

(c) If $L_k \geq 0$ and $L_{k+1} \leq 0$ for $k = 2, 3, \ldots, j-2$,

$$\begin{aligned} \mid c_j \mid \leq & \frac{1}{\left[1-\lambda+[j]_q\lambda\right]^m ([j]_q - 1)([k]_q - 1)!\Gamma_j} \\ & \times \prod_{n=2}^{k+1} \frac{\mid (A-B)(1+q)\gamma - ([n-1]_q - 1)\left[B(1+q)+(1-q)\right] \mid}{2} \end{aligned} \tag{10}$$

The bounds in (8) and (9) are sharp for all admissible A, B, $\gamma \in \mathbb{C}\setminus\{0\}$ and for each j.

Proof. Since $f(z) \in \mathcal{S}_{q,\alpha}^{\lambda,m}(\gamma; a_1, b_1; A, B)$, the inequality (6) gives

$$\begin{aligned} 2\left[zD_q\left[\mathcal{J}_\lambda^m(a_1,b_1;\,q,\,z)f\right] - \mathcal{J}_\lambda^m(a_1,b_1;\,q,\,z)f\right] = \{(A-B)\gamma(1+q)\mathcal{J}_\lambda^m(a_1,b_1;\,q,\,z)f \\ + \left[B(1+q)+(1-q)\right]\left[\mathcal{J}_\lambda^m(a_1,b_1;\,q,\,z)f - zD_q\left(\mathcal{J}_\lambda^m(a_1,b_1;\,q,\,z)f\right)\right]\}\, w(z). \end{aligned} \tag{11}$$

Equation (11) may be written as

$$\begin{aligned} \sum_{n=2}^{\infty} 2\Omega_n\left([n]_q - 1\right)\ c_n z^n = \Big\{ & (A-B)\gamma(1+q)z \\ + \sum_{n=2}^{\infty} & \left[(A-B)\gamma(1+q) - \{B(1+q)+(1-q)\}\left([n]_q - 1\right)\right]\Omega_n\, c_n z^n \Big\} w(z), \end{aligned} \tag{12}$$

where $\Omega_n = \left[1-\lambda+\lambda[n]_q\right]^m \Gamma_n$. Equivalently

$$\begin{aligned} \sum_{n=2}^{j} 2\Omega_n\left([n]_q - 1\right)\ c_n z^n + \sum_{n=j+1}^{\infty} d_n z^n = \Big\{ & (A-B)\gamma(1+q)z \\ + \sum_{n=2}^{j-1} & \left[(A-B)\gamma(1+q) - \{B(1+q)+(1-q)\}\left([n]_q - 1\right)\right]\Omega_n\, c_n z^n \Big\}\, w(z), \end{aligned} \tag{13}$$

for certain coefficients d_n. Since $| w(z) |< 1$, we have

$$\left|\sum_{n=2}^{j} 2\Omega_n \left([n]_q - 1\right) c_n z^n + \sum_{n=j+1}^{\infty} d_n z^n\right| \leq \left|(A-B)\gamma(1+q)z \right.$$
$$\left. + \sum_{n=2}^{j-1} \left[(A-B)\gamma(1+q) - \{B(1+q)+(1-q)\}\left([n]_q - 1\right)\right] \Omega_n \, c_n z^n \right|. \tag{14}$$

Let $z = re^{i\theta}$, $r < 1$, applying the Parseval's formula (see [3] p.138) on both sides of the above inequality and after simple computation , we get

$$\sum_{n=2}^{j} 4\Omega_n^2 \left([n]_q - 1\right)^2 |c_n|^2 r^{2n} + \sum_{n=j+1}^{\infty} |d_n|^2 r^{2n} \leq (A-B)^2(1+q)^2 \mid \gamma \mid^2 r^2$$

$$+ \sum_{n=2}^{j-1} \left|(A-B)\gamma(1+q) - \{B(1+q)+(1-q)\}\left([n]_q - 1\right)\right|^2 \Omega_n^2 \, |c_n|^2 r^{2n}.$$

Let $r \longrightarrow 1^-$, then on some simplification we obtain

$$4\Omega_j^2 \left([j]_q - 1\right)^2 |c_j|^2 \leq (A-B)^2(1+q)^2 \mid \gamma \mid^2 + \sum_{n=2}^{j-1} \left\{ |(A-B)\gamma(1+q) - \right.$$
$$\left. \{B(1+q)+(1-q)\}\left([n]_q - 1\right)|^2 - 4\left([n]_q - 1\right)^2 \right\} \Omega_n^2 \, |c_n|^2 \quad j \geq 2. \tag{15}$$

For $j = 2$, it follows from (15) that

$$|c_2| \leq \frac{(A-B)(1+q) \mid \gamma \mid}{2\left[1 - \lambda + [2]_q \lambda\right]^m \left([2]_q - 1\right) \Gamma_2}. \tag{16}$$

Clearly, if $L_n \geq 0$ then $L_{n-1} \geq 0$ for $n = 2, 3, \ldots$. Also, if $L_n \leq 0$ the $L_{n+1} \leq 0$ for $n = 2, 3, \ldots,$. If $Ł_2 \leq 0$, then from the above discussion we can conclude that $Ł_n \leq 0$ for all $n > 2$. It follows from (15) that

$$|c_j| \leq \frac{(A-B)(1+q) \mid \gamma \mid}{2\left[1 - \lambda + [j]_q \lambda\right]^m \left([j]_q - 1\right) \Gamma_j}. \tag{17}$$

If $L_{n-1} \geq 0$, then from the above observation $L_2, L_3, \ldots, L_{j-2} \geq 0$. From (16), we infer that the inequality (9) is true for $j = 2$. We establish (9), by mathematical induction. Suppose (9) is valid for $n = 2, 3, \ldots, (j-1)$. Then it follows from (15) that

$$4\Omega_j^2\left([j]_q-1\right)^2|c_j|^2\leq(A-B)^2(1+q)^2\mid\gamma\mid^2+\sum_{n=2}^{j-1}\{|(A-B)\gamma(1+q)- \{B(1+q)+(1-q)\}\left([n]_q-1\right)|^2-4\left([n]_q-1\right)^2\Big\}\,\Omega_n^2\,|c_n|^2$$

$$\leq(A-B)^2(1+q)^2\mid\gamma\mid^2 +\sum_{n=2}^{j-1}\Big\{|(A-B)\gamma(1+q)-\{B(1+q)+(1-q)\}\left([n]_q-1\right)|^2-4\left([n]_q-1\right)^2\Big\} \times\Omega_n^2\left\{\frac{1}{\Omega_n^2}\prod_{j=2}^{n}\frac{\mid(A-B)(1+q)\gamma-\left([j-1]_q-1\right)\left[B(1+q)+(1-q)\right]\mid^2}{2\left([j]_q-1\right)^2}\right\}$$

Thus, applying Lemma 1, we get

$$\mid c_j\mid\leq\frac{1}{[1-\lambda+[j]_q\lambda]^m\,\Gamma_j} \times\prod_{n=2}^{j}\frac{\mid(A-B)(1+q)\gamma-\left([n-1]_q-1\right)\left[B(1+q)+(1-q)\right]\mid}{2\left([n]_q-1\right)},$$

which completes the proof of (9).

Now if we assume that $L_k\geq0$ and $L_{k+1}\leq0$ for $k=2,\,3,\,\ldots,\,j-2$. Then $L_2,\,L_3,\,\ldots,\,L_{k-1}\geq0$ and $L_{k+2},\,L_{k+3},\,\ldots,\,L_{j-2}\leq0$. Then (15) gives

$$4\Omega_j^2\left([j]_q-1\right)^2\,|c_j|^2\leq(A-B)^2(1+q)^2\mid\gamma\mid^2+\sum_{l=2}^{k}\{|(A-B)\gamma(1+q) -\{B(1+q)+(1-q)\}\left([l]_q-1\right)|^2-4\left([l]_q-1\right)^2\Big\}\,\Omega_l^2\,|c_l|^2 +\sum_{l=k+1}^{j-1}\Big\{[(A-B)(1+q)\gamma-[B(1+q)+(1-q)]\left([l]_q-1\right)]^2 -4\left([l]_q-1\right)^2\Big\}\,\Omega_l^2\,|c_l|^2$$

$$\leq(A-B)^2(1+q)^2\mid\gamma\mid^2 +\sum_{l=2}^{k}\Big\{[(A-B)(1+q)\gamma-[B(1+q)+(1-q)]\left([l]_q-1\right)]^2 -4\left([l]_q-1\right)^2\Big\}\,\Omega_l^2\,|c_l|^2\,.$$

On substituting upper estimates for $c_2,\,c_3,\,\ldots,\,c_k$ obtained above and simplifying, we obtain (10).

Also, the bounds in (8) are sharp for the functions $f_k(z)$ given by

$$\mathfrak{I}_\lambda^m(a_1,b_1;\,q,\,z)f_k(z)=\begin{cases}z(1+Bz)^{\frac{(A-B)\gamma}{B\lambda(k-1)}} & if\ B\neq0,\\ z\,exp\left(\frac{A\gamma}{\lambda(k-1)}\,z^{k-1}\right) & if\ B=0.\end{cases}$$

The bounds in (9) are sharp for the functions $f(z)$ given by

$$\mathfrak{J}_{\lambda}^{m}(a_1, b_1;\, q,\, z)f = \begin{cases} z(1+Bz)^{\frac{(A-B)\gamma}{B}} & if\, B \neq 0, \\ z\, exp(A\gamma z) & if\, B = 0. \end{cases}$$

□

Remark 4. *It can be seen that Theorem 1 extends the results obtained by various authors, for example see [4, 19, 22, 23].*

If we let $m = 1$, $r = 2$, $s = 1$, $a_1 = b_1$, $a_2 = q$ and $q \to 1^-$, we get the following result obtained by Ghosh and Vasudevarao in [4].

Corollary 1. *Let $f \in \mathcal{A}$ satisfy the condition*

$$1 + \frac{1}{\gamma}\left(\frac{z\left[\lambda z f'(z) + (1-\lambda)f(z)\right]'}{\lambda z f'(z) + (1-\lambda)f(z)} - 1\right) \prec \frac{1+Az}{1+Bz}.$$

Also, let $L_n = |(A-B)\gamma - B\,(n-1)| - 4\,(n-1)$.

(a) If $L_2 \leq 0$, then

$$|c_j| \leq \frac{(A-B)\mid \gamma \mid}{[1+\lambda(j-1)]\,(j-1)}.$$

(b) If $L_n \geq 0$, then

$$\mid c_j \mid \leq \frac{1}{[1+\lambda(j-1)]}\prod_{n=2}^{j}\frac{\mid (A-B)\gamma - (n-2)\,B \mid}{(n-1)}$$

(c) If $L_k \geq 0$ and $L_{k+1} \leq 0$ for $k = 2, 3, \ldots, j-2$,

$$\mid c_j \mid \leq \frac{1}{[1+\lambda(j-1)]\,(j-1)\,(k-1)!}\prod_{n=2}^{k+1}\mid (A-B)\gamma - (n-2)\,B \mid$$

The bounds in (a) and (b) are sharp for all admissible A, B, $\gamma \in \mathbb{C} \setminus \{0\}$ and for each j.

Corollary 2. *[23] Let $f \in \mathcal{A}$, satisfy the subordination condition*

$$(1 + i\tan\beta)\,\frac{zf'(z)}{f(z)} - i\tan\beta \prec \frac{1+Az}{1+Bz}, \qquad (z \in \mathbb{U}).$$

Then for $n \geq 2$,

$$|c_n| \leq \prod_{j=0}^{n-2}\frac{\left|(A-B)e^{-i\beta}\cos\beta - jB\right|}{(j+1)}$$

where $|\beta| < \pi/2$.

If we set $A = 1 - 2\eta\,(0 \leq \eta < 1)$, $B = -1$, $m = 0$ and $q \to 1^-$ in Corollary 2, we get the following result.

Corollary 3. *[11] Let $f \in \mathcal{A}$ satisfy the condition*

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \eta\cos\beta.$$

Then

$$|c_n| \leq \prod_{j=0}^{n-2}\left(\frac{|2(1-\eta)e^{-i\beta}\cos\beta + j|}{j+1}\right). \tag{18}$$

The coefficient estimates in (18) are sharp.

If we let $\lambda = 1$, $r = 2$, $s = 1$, $a_1 = b_1$ and $a_2 = q$, $A = 1$, $B = -M$ and $q \to 1^-$ in Theorem 1, we have

Corollary 4. *[1] Let the function $f(z)$ defined by (1) satisfy the condition*

$$\left|\frac{\gamma - 1 + \frac{\mathscr{D}^{m+1}f(z)}{\mathscr{D}^m f(z)}}{\gamma} - M\right| < M.$$

Let

$$G = \left[\frac{2u(n-1)\,Re\,(\gamma)}{(n-1)^2(1-u) - |\gamma|^2(1+u)}\right],$$

$$(\text{for } n = 1,\ 3,\ \ldots,\ j-1).$$

(a) If $2u(n-1)Re\,\{\gamma\} > (n-1)^2(1-u) - |\,\gamma\,|^2\,(1+u)$, then, for $j = 2, 3, \ldots, G+2$

$$|\,c_j\,| \leq \frac{1}{j^m\,(j-1)!}\prod_{n=2}^{j}|\,(1+u)\gamma + (n-2)u\,| \tag{19}$$

and for $j > G+2$

$$|\,c_j\,| \leq \frac{1}{j^m(j-1)\,(G+1)!}\prod_{n=2}^{G+3}|\,(1+u)\gamma + (n-2)u\,|$$

(b) If $2u(n-1)\,Re\,\{\gamma\} \leq (n-1)^2(1-u) - |\gamma|^2(1+u)$, then

$$|\,c_j\,| \leq \frac{(1+u)\,|\,\gamma\,|}{(j-1)\,j^m} \quad j \geq 2. \tag{20}$$

where $u = 1 - \dfrac{1}{M}\left(M > -\dfrac{1}{2}\right)$.
The inequalities (19) and (20) are sharp.

To prove our next result, we need the following Lemmas.

Lemma 2. *[16] Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ be analytic in $\mathbb{U}$ and $g(z) = \sum_{n=1}^{\infty} b_z z^n$ is analytic and convex in $\mathbb{U}$. If $f(z) \prec g(z)$, then $|a_n| \leq |b_1|$, for $n = 1, 2, \ldots$.*

Lemma 3. *Let the function $p_\alpha(z)$ be defined as in (2) and let $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ be in $\mathcal{P}$ satisfy the condition*

$$p(z) \prec \frac{(A+1)p_\alpha(z) - (A-1)}{(B+1)p_\alpha(z) - (B-1)}. \tag{21}$$

Then

$$|p_n| \leq \frac{(A-B)(1+4\alpha)}{2(1+2\alpha)}, \qquad (n \geq 1). \tag{22}$$

Proof. From (21), we have

$$\begin{aligned} p(z) &\prec \frac{A-1}{B-1}\left[1 - \frac{A+1}{A-1}p_\alpha(z)\right]\left[1 + \frac{B+1}{B-1}p_\alpha(z) + \left(\frac{B+1}{B-1}p_\alpha(z)\right)^2 + \cdots\right] \\ &= \frac{A-1}{B-1}\left[1 + \left(\frac{B+1}{B-1} - \frac{A+1}{A-1}\right)p_\alpha(z)\right. \\ &\qquad \left. + \left(\frac{(B+1)^2}{(B-1)^2} - \frac{(A+1)(B+1)}{(A-1)(B-1)}\right)[p_\alpha(z)]^2 + \cdots\right]. \end{aligned}$$

Using $p_\alpha(z) = 1 + L_1 z + L_2 z^2 + L_3 z^3 + L_4 z^4 + \cdots$ in the above condition and simplifying, we get

$$p(z) \prec \sum_{n=1}^{\infty} \frac{-2(B+1)^{n-1}}{(B-1)^n} + \left[\sum_{n=1}^{\infty} \frac{2n(A-B)(B+1)^{n-1}}{(B-1)^{n+1}}\right] L_1 z + \cdots.$$

The series $\sum_{n=1}^{\infty} \frac{-2(B+1)^{n-1}}{(B-1)^n}$ and $\sum_{n=1}^{\infty} \frac{2n(A-B)(B+1)^{n-1}}{(B-1)^{n+1}}$ converges to 1 and $\frac{(A-B)}{2}$ respectively. Hence

$$p(z) \prec 1 + \frac{(A-B)(1+4\alpha)}{2(1+2\alpha)} z + \cdots. \tag{23}$$

Since $p(z) \in \mathcal{P}$ and the superordinate function in (23) is convex in $\mathbb{U}$, the result follows on applying Lemma 2 to (23). □

Theorem 2. *Let $f \in k - \mathcal{SH}_{q,\alpha}^{\lambda, m}(\gamma; a_1, b_1; A, B)$, then for $n \geq 2$,*

$$|c_n| \leq \frac{1}{[1 - \lambda + \lambda[n]_q]^m \Gamma_n} \left(\prod_{j=0}^{n-2} \frac{|(A-B)(1+4\alpha)\gamma - 2(1+2\alpha)([j+1]_q - 1)B|}{2(1+2\alpha)([j+2]_q - 1)} \right). \tag{24}$$

Proof. By the definition of $k - \mathcal{SH}_{q,\alpha}^{\lambda;\,m}(\gamma; a_1,\, b_1; A, B)$, we have

$$1 + \frac{1}{\gamma}\left(\frac{zD_q\left[\mathcal{J}_\lambda^m(a_1, b_1;\, q,\, z)f\right]}{\mathcal{J}_\lambda^m(a_1, b_1;\, q,\, z)f} - 1\right) = p(z), \tag{25}$$

where $p(z) \in \mathcal{P}$ and satisfies the subordination condition

$$p(z) \prec \frac{(A+1)p_\alpha(z) - (A-1)}{(B+1)p_\alpha(z) - (B-1)}.$$

Equivalently (25) can be rewritten as

$$\sum_{n=2}^{\infty} \left[1 - \lambda + \lambda[n]_q\right]^m \left([n]_q - 1\right)\, \Gamma_n c_n z^n$$
$$= \gamma\left[z + \sum_{n=2}^{\infty} \left[1 - \lambda + \lambda[n]_q\right]^m\, \Gamma_n c_n z^n\right]\left[\sum_{n=1}^{\infty} p_n z^n\right].$$

On equating the coefficient of z^n, we get

$$\left[1 - \lambda + \lambda[n]_q\right]^m \left([n]_q - 1\right)\, \Gamma_n c_n$$
$$= \gamma\left\{p_{n-1} + p_{n-2}\left[1 - \lambda + \lambda[2]_q\right]^m \Gamma_2 c_2 + \cdots + p_1\left[1 - \lambda + \lambda[n-1]_q\right]^m \Gamma_{n-1} c_{n-1}\right\}.$$

On computation, we have

$$|c_n| \leq \frac{|\gamma|}{\left[1 - \lambda + \lambda[n]_q\right]^m \left([n]_q - 1\right)\Gamma_n} \sum_{j=1}^{n-1} \left[1 - \lambda + \lambda[n-j]_q\right]^m \Gamma_{n-j}|c_{n-j}||p_j|.$$

Using (22) in the above inequality, we have (for $c_1 = 1$)

$$|c_n| \leq \frac{|\gamma|(A-B)(1+4\alpha)}{2(1+2\alpha)\left[1 - \lambda + \lambda[n]_q\right]^m \left([n]_q - 1\right)\Gamma_n} \sum_{j=1}^{n-1} \left[1 - \lambda + \lambda[n-j]_q\right]^m \Gamma_{n-j}|c_{n-j}|. \tag{26}$$

Taking $n = 2$, in (26), we get

$$|c_2| \leq \frac{|\gamma|(A-B)(1+4\alpha)}{2q(1+2\alpha)\left[1 - \lambda + \lambda[2]_q\right]^m \Gamma_2}$$

and for $n = 3$ in (26), we get

$$|c_3| \leq \frac{|\gamma|(A-B)(1+4\alpha)}{2q(1+q)(1+2\alpha)\left[1 - \lambda + \lambda[3]_q\right]^m \Gamma_3}\left(1 + \left[1 - \lambda + \lambda[2]_q\right]^m \Gamma_2|c_2|\right)$$
$$\leq \frac{|\gamma|(A-B)(1+4\alpha)}{2q(1+q)(1+2\alpha)\left[1 - \lambda + \lambda[3]_q\right]^m \Gamma_3}\left[1 + \frac{|\gamma|(A-B)(1+4\alpha)}{2q(1+2\alpha)}\right].$$

If we let $n = 3$, in (24), we have

$$\begin{aligned} |c_3 &\leq \frac{1}{\left[1-\lambda+\lambda[3]_q\right]^m \Gamma_3}\left[\frac{|\gamma|(A-B)(1+4\alpha)}{2q(1+2\alpha)}\right. \\ &\qquad\qquad \left.\times\frac{|(A-B)\gamma(1+4\alpha)-2(1+2\alpha)(q)B|}{2(1+2\alpha)q(1+q)}\right] \\ &\leq \frac{1}{\left[1-\lambda+\lambda[3]_q\right]^m \Gamma_3}\left[\frac{|\gamma|(A-B)(1+4\alpha)}{2q(1+2\alpha)}\right. \\ &\qquad\qquad \left.\times\frac{(A-B)|\gamma|(1+4\alpha)+2(1+2\alpha)(q)|B|}{2(1+2\alpha)q(1+q)}\right] \\ &\leq \frac{|\gamma|(A-B)(1+4\alpha)}{2q(1+q)(1+2\alpha)\left[1-\lambda+\lambda[3]_q\right]^m \Gamma_3}\left[\frac{|\gamma|(A-B)(1+4\alpha)}{2q(1+2\alpha)}+1\right]. \end{aligned}$$

Hence the hypothesis of the theorem is true for $n = 3$.

Now let us suppose (24) is valid for $n = 2, 3, \ldots r$. On using triangle inequality in (24), we get

$$\begin{aligned} |c_r| &\leq \frac{1}{\left[1-\lambda+\lambda[r]_q\right]^m \left([r]_q-1\right)\Gamma_r} \\ &\quad\times\left(\prod_{j=0}^{r-2}\frac{(A-B)(1+4\alpha)|\gamma|+2(1+2\alpha)([j+1]_q-1)}{2(1+2\alpha)([j+2]_q-1)}\right). \end{aligned}$$

By induction hypothesis, we have

$$\begin{aligned} &\frac{|\gamma|(A-B)(1+4\alpha)}{2(1+2\alpha)\left[1-\lambda+\lambda[r]_q\right]^m \left([r]_q-1\right)\Gamma_r}\sum_{j=1}^{r-1}\left[1-\lambda+\lambda[r-j]_q\right]^m \Gamma_{r-j}|c_{r-j}| \\ &\leq \frac{1}{\left[1-\lambda+\lambda[r]_q\right]^m \Gamma_r}\prod_{j=0}^{r-2}\frac{(A-B)(1+4\alpha)|\gamma|+2(1+2\alpha)([j+1]_q-1)}{2(1+2\alpha)([j+2]_q-1)}. \end{aligned}$$

From the above inequality, we have

$$\begin{aligned} &\prod_{j=0}^{r-1}\frac{(A-B)(1+4\alpha)|\gamma|+2(1+2\alpha)([j+1]_q-1)}{2(1+2\alpha)([j+2]_q-1)\left[1-\lambda+\lambda[r+1]_q\right]^m \Gamma_{r+1}} \\ &\geq \frac{|\gamma|(A-B)(1+4\alpha)}{2(1+2\alpha)\left([r]_q-1\right)}\times\frac{(A-B)(1+4\alpha)|\gamma|+2(1+2\alpha)([r]_q-1)}{2(1+2\alpha)([r+1]_q-1)\left[1-\lambda+\lambda[r+1]_q\right]^m \Gamma_{r+1}} \\ &\qquad\times\sum_{j=1}^{r-1}\left[1-\lambda+\lambda[r-j]_q\right]^m \Gamma_{r-j}|c_{r-j}| \\ &=\frac{|\gamma|(A-B)(1+4\alpha)}{2(1+2\alpha)\left[1-\lambda+\lambda[r+1]_q\right]^m \left([r+1]_q-1\right)\Gamma_{r+1}} \\ &\qquad\left[|c_r|\left[1-\lambda+\lambda[r]_q\right]^m \Gamma_r+\sum_{j=1}^{r-1}\left[1-\lambda+\lambda[r-j]_q\right]^m \Gamma_{r-j}|c_{r-j}|\right] \end{aligned}$$

$$= \frac{|\gamma|(A-B)(1+4\alpha)}{2(1+2\alpha)\left[1-\lambda+\lambda[r+1]_q\right]^m\left([r+1]_q-1\right)\Gamma_{r+1}} \times \sum_{j=1}^{r}\left[1-\lambda+\lambda[r+1-j]_q\right]^m \Gamma_{r+1-j}|c_{r+1-j}|,$$

implies that inequality is true for $n = r+1$. Hence the proof of the Theorem. □

If we choose $\gamma = 1/(1+i\tan\beta)$, $\lambda = 1$, $r = 2$, $s = 1$, $a_1 = b_1$, $a_2 = q$, $m = 0$ and $q \to 1^-$ in Theorem 2, we get the following

Corollary 5. *Let $f \in \mathcal{A}$, satisfy the subordination condition*

$$(1+i\tan\beta)\frac{zf'(z)}{f(z)} - i\tan\beta \prec \frac{(A+1)p_\alpha(z)-(A-1)}{(B+1)p_\alpha(z)-(B-1)}, \qquad (z \in \mathbb{U}).$$

Then for $n \geq 2$,

$$|c_n| \leq \prod_{j=0}^{n-2} \frac{\left|(A-B)(1+4\alpha)e^{-i\beta}\cos\beta - jB\right|}{2(1+2\alpha)(j+1)}$$

where $|\beta| < \pi/2$.

If we choose $\gamma = 1+i0$, $m = 0$, $r = 2$, $s = 1$, $a_1 = b_1$, $a_2 = q$ and $q \to 1^-$ in Theorem 2, we get the following, (for an analogous result see [12]).

Corollary 6. *For a function $p_\alpha(z)$ defined as in (2). Let $f \in \mathcal{A}$ satisfy the condition*

$$\frac{zf'(z)}{f(z)} \prec \frac{(A+1)p_\alpha(z)-(A-1)}{(B+1)p_\alpha(z)-(B-1)}$$

then

$$|c_n| \leq \prod_{j=0}^{n-2} \frac{|(A-B)(1+4\alpha)-2jB|}{2(1+2\alpha)(j+1)}.$$

If we choose $m = \lambda = 1$, $r = 2$, $s = 1$, $a_1 = b_1$, $a_2 = q$ and $q \to 1^-$ in Theorem 2, we get the following.

Corollary 7. *For a function $p_\alpha(z)$ defined as in (2). Let $f \in \mathcal{A}$ satisfy the condition*

$$1+\frac{1}{\gamma}\left(\frac{zf''(z)}{f'(z)}\right) \prec \frac{(A+1)p_\alpha(z)-(A-1)}{(B+1)p_\alpha(z)-(B-1)}$$

then

$$|c_n| \leq \frac{1}{n}\prod_{j=0}^{n-2} \frac{|(A-B)(1+4\alpha)-2jB|}{2(1+2\alpha)(j+1)}.$$

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