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ON INVARIANT SUBMANIFOLDS OF PARACONTACT (κ, μ)-SPACES

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Abstract

The object of the present paper is to deduce some necessary and sufficient conditions for invariant Submanifolds of paracontact (κ, μ)-spaces to be totally geodesic. We also establish that a totally umbilical invariant submanifold of a paracontact (κ, μ)-manifold is also totally geodesic. Some more necessary and sufficient conditions for a submanifold of a paracontact (κ, μ)-manifold to be totally geodesic have been deduced using parallelity and pseudo parallelity of the second fundamental form. In the last section we obtain some results on paracontact (κ, μ)-manifold with concircular canonical field.

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1 Introduction

Submanifold theory is a major branch of differential geometry and plays an important role in the development of the subject. Among all submanifolds, invariant submanifolds are very interesting and useful. The study of invariant submanifolds is a growing topic of research in differential geometry. B. Y. Chen [4],[5] has done many works in this line. Invariant submanifolds has been studied by several authors [8],[11]. Paracontact[1] metric manifolds have become a thurst of research in the field of metric geometry. Recently, Cappelletti-Montano et. al. [2] introduced a new type of paracontact geometry the so-called para-contact metric (κ, μ) spaces, where κ and μ are real constants.

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In modern analysis, the geometry of submaniofolds have become a subject of growing interest for its significant applications in applied mathematics and theoretical physics. For instance, the notion of invariant submanifold is used to discuss properties of non-linear autonomous system. Also, the notion of geodesic plays an important role in the theory of relativity. For totally geodesic submanifolds, the geodesics of the ambient manifolds remain geodesics in the submanifolds. Hence, totally geodesic submanifolds have also importance in physical sciences.There has been several papers on Riemannian manifolds which admit concircular vector fields and also concurrent vector fields. Recently B. Y. Chen and S. W. Wei studied Riemannian submanifolds with concircular canonical vector field in [3].

The present paper is organized as follows: After the introduction in Section 1, we give the required preliminaries in Section 2. In Section 3 we show that if an odd-dimensional submanifold of a paracontact (κ, μ)-manifold is invariant then it is totally geodesic. The converse is also true. In Section 4 we give an example. Recurrent submanifolds of paracontact metric (κ, μ) manifolds have been studied in Section 5. Totally umbilical submanifolds of a paracontact (κ, μ)-manifolds have been studied in Section 6. The Section 7 contains the study of submanifolds whose second fundamental forms satisfy some parallelity and pseudo symmetry conditions. In the last section we establish some results on submanifolds of a paracontact metric (κ, μ) manifolds with concircular canonical field.

2 Preliminaries

A smooth manifold $\widetilde{M}^{2n+1}$ is said to admit an almost para contact structure (ϕ, ξ, η) if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying
$\phi^2 X = X - \eta(X)\xi, \quad \eta(\xi) = 1$, for any vector field $X \in \chi(\widetilde{M})$, the set of all differentiable vector fields on $\widetilde{M}$. For such manifolds we also have
$(i) \quad \phi(\xi) = 0, \qquad \eta \circ \phi = 0,$
(ii) the tensor field ϕ induces an almost paracontact structure on each fibre of $D = \ker(\eta)$, that is the eigen distributions D^+_ϕ and D^-_ϕ of ϕ corresponding to the eigen values 1 and -1, respectively, have same dimension n.

An almost paracontact structure is said to be normal if and only if the (1,2)-type torsion tensor $N_\phi = [\phi, \phi] - 2d\eta \otimes \xi$ vanishes identically, where $[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$. An almost paracontact manifold equipped with a pseudo-Riemannian metric g such that

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y) \tag{1}$$

for all $X, Y \in \chi(M)$, is called almost paracontact metric manifold, where signature of g is $(n+1, n)$. An almost paracontact structure is said to be a paracontact structure if $g(X, \phi Y) = d\eta(X, Y)$ with the associated metric g. For an almost paracontact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ admits a ϕ-basis, that is a pseudo-orthonormal basis of vector fields of the form $\{\xi, E_1, E_2, ..., E_n, \phi E_1, \phi E_2, ..., \phi E_n\}$, where $\xi, E_1, E_2, ..E_n$ are space-like vector fields and the vector fields $\phi E_1, \phi E_2, ..\phi E_n$ are time-like.

In a paracontact metric manifold we define a symmetric, trace-free (1,1)-tensor field $h = \frac{1}{2}\mathcal{L}_\xi \phi$ satisfying

$$\phi h + h\phi = 0, h\xi = 0, \tag{2}$$

$$\widetilde{\nabla}_X \xi = -\phi X + \phi h X, \text{for all } X \in \chi(\widetilde{M}), \tag{3}$$

where $\widetilde{\nabla}$ is the Levi-Civita connection of the pseudo-Riemannian manifold. Noticing that the tensor h vanishes identically if and only if ξ is a killing vector field and in such case (ϕ, ξ, η, g) is said to be a K-paracontact structure. An almost paracontact manifold is said to be para-Sasakian if and only if the following condition holds

$$(\widetilde{\nabla}_X \phi)Y = -g(X, Y)\xi + \eta(Y)X \tag{4}$$

for any $X, Y \in \chi(M)$. A normal paracontact metric manifold is para-Sasakian and satisfies

$$R(X, Y)\xi = -(\eta(Y)X - \eta(X)Y) \tag{5}$$

for any $X, Y \in \chi(M)$, but unlike contact metric geometry (5) is not a sufficient condition for a paracontact manifold to be para-Sasakian. It is clear that every para-Sasakian manifold is K-paracontact, but the converse is not always true.

Finally, we recall the definition of paracontact metric (κ, μ)-manifolds:

A paracontact metric manifold is said to be a paracontact (κ, μ)-manifold if the curvature tensor $\widetilde{R}$ satisfies

$$\widetilde{R}(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \tag{6}$$

for all vector fields $X, Y \in \chi(M)$ and κ, μ are real constants.

This class is very wide containing the para-Sasakian manifolds as well as paracontact metric manifolds satisfying $\widetilde{R}(X, Y)\xi = 0$ for all $X, Y \in \chi(M)$.

In particular, if $\mu = 0$, then the paracontact metric (κ, μ)-manifold is called $N(\kappa)$-paracontact metric manifold. Thus for a $N(\kappa)$-paracontact metric manifold the curvature tensor satisfies the following relation

$$\widetilde{R}(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) \tag{7}$$

for all $X, Y \in \chi(M)$.

In a paracontact metric (κ, μ)-manifold $(M^{2n+1}, \phi, \xi, \eta, g), n \geq 1$, the following relations hold [2]:

$$h^2 = (\kappa + 1)\phi^2, \tag{8}$$

$$(\nabla_X \phi)Y = -g(X - hX, Y)\xi + \eta(Y)(X - hX) \tag{9}$$

for $\kappa \neq -1$.

Let M be a submanifold immersed in an n-dimensional pseudo-Riemannian manifold $\widetilde{M}$. We denote by the same symbol g the induced metric on M. Let TM be the tangent bundle of M and $T^{\perp}M$ is the set of all vector fields normal to M. Then Gauss and Weingarten formulae are given by [2]

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \tag{10}$$

$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \tag{11}$$

for any tangent vector fields X, Y of M and normal vector fields N of M, where $\nabla^{\perp}$ is the connection in the normal bundle. The second fundamental form σ and A_N are related by

$$g(A_N X, Y) = g(\sigma(X,Y), N). \tag{12}$$

It is also noted that $\sigma(X,Y)$ is bilinear, and since $\nabla_{fX}\mathrm{Y}=f\nabla_X\mathrm{Y}$, for a C^{∞} function f on a manifold we have

$$\sigma(fX, Y) = f\sigma(X,Y). \tag{13}$$

Let us now recall the following:

Definition 1. *Let, $\widetilde{M}$ be an n-dimensional paracontact (κ,μ)-spaces and M be a submanifold of $\widetilde{M}$. The submanifold M of $\widetilde{M}$ is said to be invariant if the structure vector field ξ is tangent to M, at every point of M and ϕX is tangent to M for any vector field X tangent to M, at every point on M, that is, $\phi TM \subset TM$ at every point on M.*

Definition 2. *A submanifold of a paracontact (κ,μ)-spaces is called totally geodesic if $\sigma(X,Y) = 0$, for any $X, Y \in TM$.*

Definition 3. *The second fundamental form σ is said to be recurrent, respectively, 2-recurrent if the following conditions hold :*

$$(\nabla_W \sigma)(Y,Z) = A(W)\sigma(Y,Z), \tag{14}$$

$$(\nabla_U \nabla_W \sigma)(Y,Z) = B(U,W)\sigma(Y,Z), \tag{15}$$

where A is a 1-form on M and B is 2-form on M.

3 Invariant submanifolds of paracontact metric (κ,μ) manifolds

Proposition 1. *Let, M be an invariant submanifold of a paracontact metric (κ,μ) manifold. Then there exists two differentiable orthogonal distributions D and $D^{\perp}$ on M such that*

$$TM = D \oplus D^{\perp} \oplus <\xi>, \qquad \textit{and} \qquad \phi(D) \subset D^{\perp},\ \phi(D^{\perp}) \subset D.$$

Proof. The proof is similar to the analogue proof in [13]. $\square$

Proposition 2. *For an invariant submanifold M of a paracontact metric (κ,μ) manifold, for the two differentiable tangent vector fields X, Y of M, we have*
$\sigma(X,\xi) = 0$,
$\sigma(X,\phi Y) = \phi\sigma(X,Y) = \sigma(\phi X,Y)$.

Proof. The proof is similar to the analogue proof in [13]. $\square$

Theorem 1. *Every odd dimensional invariant submanifold of a paracontact metric (κ,μ) manifold is totally geodesic.*

Proof. The proof is similar to the analogue proof in [13]. $\square$

Now we shall show that the converse is true irrespective of dimension.

Theorem 2. *Every totally geodesic submanifold of a paracontact metric (κ,μ) manifold is invariant.*

Proof. Let, the submanifold be totally geodesic. So, $\sigma(X,Y)$=0 for $X,Y \in TM$. Now we know that

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y). \tag{16}$$

Putting $Y = \xi$, we have from above

$$\widetilde{\nabla}_X \xi = \nabla_X \xi + \sigma(X,\xi). \tag{17}$$

Again, for paracontact metric (κ,μ) manifold, we get

$$\widetilde{\nabla}_X \xi = -\phi X + \phi h X. \tag{18}$$

Since σ(X,ξ)=0, then from (17) and (18) we get

$$-\phi X + \phi h X = \nabla_X \xi. \tag{19}$$

From (19) it is clear that ϕX$\in TM$. So, the submanifold is invariant. $\square$

Theorem 3. *An invariant submanifold of a paracontact metric (κ,μ) manifold is also a paracontact metric (κ,μ) manifold provided it is odd dimensional.*

Proof. Let $\widetilde{M}$ be a paracontact metric (κ,μ) manifold. And also let, M be an invariant submanifold of $\widetilde{M}$.
We shall prove that, M is also a paracontact metric (κ,μ) manifold.
Since, $\widetilde{M}$ is a paracontact metric (κ,μ) manifold, we get

$$(\widetilde{\nabla}_X \phi)Y = -g(X - hX, Y)\xi + \eta(Y)(X - hX), \tag{20}$$

where $\widetilde{\nabla}$ is the Levi-Civita connection of the pseudo-Riemannian manifold.
By covariant differentiation, we get

$$(\widetilde{\nabla}_X \phi)Y = \widetilde{\nabla}_X \phi Y - \phi(\widetilde{\nabla}_X Y). \tag{21}$$

Again, by (10)

$$\widetilde{\nabla}_X \phi Y = \nabla_X \phi Y + \sigma(X, \phi Y), \widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y). \tag{22}$$

Combining (21) and (22), we get,

$$(\widetilde{\nabla}_X \phi)Y = \nabla_X \phi Y + \sigma(X, \phi Y) - \phi(\nabla_X Y + \sigma(X, Y)). \tag{23}$$

Suppose the submanifold is odd dimensional. Then by Theorem (1),
$\sigma(X, Y) = 0$, for any $X, Y \in TM$.
From (23), we get

$$(\widetilde{\nabla}_X \phi)Y = \nabla_X \phi Y - \phi(\nabla_X Y) = (\nabla_X \phi)Y. \tag{24}$$

Hence, we get from (20)

$$(\nabla_X \phi)Y = -g(X - hX, Y)\xi + \eta(Y)(X - hX).$$

This shows that the invariant submanifold M is also paracontact metric (κ, μ) manifold. Hence, the theorem follows. □

4 Examples

In this section we would like to construct an example of a five-dimensional paracontact metric (κ, μ) manifold and there on an example of three-dimensional invariant submanifold of the manifold. The example is taken from [12].

Let us consider the 5-dimensional manifold $\widetilde{M}=\{(x, y, z, w, s)\in\mathbb{R}^4 | s \neq 0\}$, where (x, y, z, w, s) are the standard coordinates in $\mathbb{R}^5$.
The vector fields $\{X_1, Y_1, X_2, Y_2, \xi\}$ are linearly independent at each point of $\widetilde{M}$, such that

$$[\xi, X_1] = X_1 + Y_1,\ [\xi, Y_1] = -Y_1,\ [X_1, Y_1] = 2\xi,$$

$$[X_2, Y_2] = 2(\xi + Y_2),\ [X_1, Y_2] = X_1 + Y_1,\ [Y_1, Y_2] = -Y_1.$$

Let, g be the Pseudo-Riemannian metric defined by

$$g(\xi, \xi) = g(X_1, Y_1) = 1,\ g(X_2, Y_2) = -1,$$

all other components of the metric are zero.

Let, η be the 1-form defined by

$$\eta(Z) = g(Z,\ \xi) \text{ for any } Z\in \chi(\widetilde{M})$$

Let, ϕ be the (1,1) tensor field defined by

$$\phi(X_1) = X_1,\ \phi(Y_1) = -Y_1,\ \phi(X_2) = -X_2,\ \phi(Y_2) = Y_2,\ \phi(\xi) = 0.$$

Then using the linearity of ϕ and g we have

$$\eta(\xi) = 1, \qquad \phi^2 Z = Z - \eta(Z)\xi,$$

$$g(\phi Z, \phi W) = -g(Z, W) + \eta(Z)\eta(W) \quad \text{for any} \ \ Z, W \in \chi(\widetilde{M}).$$

Then the structure (ϕ, ξ, η, g) defines an almost para contact structure on $\widetilde{M}$.

Let, $\widetilde{\nabla}$ be the Levi-Civita connection with respect to the pseudo-Riemannian metric g.

A straight forward computation gives that $hX_1 = Y_1, hY_1 = 0$ and $hX_2 = hY_2 = 0$.

Moreover, by basic paracontact metric properties and using Koszul formula, we can easily calculate the following :

$$\widetilde{\nabla}_\xi \xi = 0,\ \widetilde{\nabla}_\xi X_1 = 0,\ \widetilde{\nabla}_\xi Y_1 = 0,\ \widetilde{\nabla}_\xi X_2 = X_2,\ \widetilde{\nabla}_\xi Y_2 = -Y_2,$$

$$\widetilde{\nabla}_{X_1} \xi = -X_1 - Y_1,\ \widetilde{\nabla}_{X_1} X_1 = 0,\ \widetilde{\nabla}_{X_1} Y_1 = \xi + 2Y_1,\ \widetilde{\nabla}_{X_1} X_2 = 0,\ \widetilde{\nabla}_{X_1} Y_2 = 0,$$

$$\widetilde{\nabla}_{Y_1} \xi = Y_1,\ \widetilde{\nabla}_{Y_1} X_1 = -\xi,\ \widetilde{\nabla}_{Y_1} Y_1 = 0,\ \widetilde{\nabla}_{Y_1} X_2 = 0,\ \widetilde{\nabla}_{Y_1} Y_2 = 0,$$

$$\widetilde{\nabla}_{X_2} \xi = X_2,\ \widetilde{\nabla}_{X_2} X_1 = 0,\ \widetilde{\nabla}_{X_2} Y_1 = 0,\ \widetilde{\nabla}_{X_2} X_2 = 0,\ \widetilde{\nabla}_{X_2} Y_2 = \xi + 2Y_2,$$

$$\widetilde{\nabla}_{Y_2} \xi = -Y_2,\ \widetilde{\nabla}_{Y_2} X_1 = 0,\ \widetilde{\nabla}_{Y_2} Y_1 = Y_1,\ \widetilde{\nabla}_{Y_2} X_2 = -\xi,\ \widetilde{\nabla}_{Y_2} Y_2 = 0,$$

and thus

$$R(X_1, \xi)\xi = -X_1, R(X_2, \xi)\xi = -X_2,$$

$$R(Y_1, \xi)\xi = -Y_1, R(Y_2, \xi)\xi = -Y_2,$$

$$R(X_i, X_j)\xi = R(X_i, Y_j)\xi = R(Y_i, Y_j)\xi = 0, \qquad i, j = 1, 2.$$

From the above it can be easily seen that $\widetilde{M}^5(\phi, \xi, \eta, g)$ is a paracontact (-1,0)-space.

Let, f be an isometric immersion from M to $\widetilde{M}$ defined by $f(x, y, z) = (x, y, z, 0, 0)$.

Let, $M = \{(x, y, z) \in \mathbb{R}^3 \mid z \neq 0\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. The vector fields $\{X_1, Y_1, \xi\}$ are linearly independent at each point of M, such that

$$[\xi, X_1] = X_1 + Y_1,\ [\xi, Y_1] = -Y_1,\ [X_1, Y_1] = 2\xi.$$

Let, g be the Pseudo-Riemannian metric defined by

$$g(\xi, \xi) = 1,\ g(X_1, Y_1) = -1,$$

all other components of the metric are zero.

Let, η be the 1-form defined by

$$\eta(Z) = g(Z, \xi) \text{ for any } Z \in \chi(M).$$

Let, ϕ be the (1,1) tensor field defined by

$$\phi(X_1) = X_1,\ \phi(Y_1) = -Y_1,\ \phi(\xi) = 0.$$

Then using the linearity of ϕ and g we have

$$\eta(\xi) = 1, \qquad \phi^2 Z = Z - \eta(Z)\xi,$$

$$g(\phi Z, \phi W) = -g(Z, W) + \eta(Z)\eta(W) \quad \text{for any} \quad Z, W \in \chi(M).$$

Then the structure (ϕ, ξ, η, g) defines an almost para contact structure on M.

A straight forward computation gives that $hX_1 = Y_1, hY_1 = 0$.
It is obvious that the manifold M under consideration is a submanifold of the manifold $\widetilde{M}$.

Remark 1. By Theorem 1, every odd dimensional invariant submanifold of a paracontact (κ, μ) manifold is totally geodesic. So, a natural question arises that whether an even dimensional invariant submanifold is totally geodesic. We answer this question in the following sections.

5 Recurrent submanifolds of paracontact metric (κ, μ) manifolds

A submanifold is called recurrent, if its second fundamental form is recurrent. To prove the main theorem, first we prove an important lemma.

Lemma 1. *Let, M be an invariant submanifold of a paracontact metric (κ, μ) manifold. Then for $X, \xi \in M$*

$$\sigma(X, \xi) = 0 \quad \text{and} \quad \nabla_X \xi = -\phi X + \phi h X.$$

Proof. From equation (10), we get

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y). \tag{25}$$

Putting, $Y{=}\xi$, we have

$$\widetilde{\nabla}_X \xi = \nabla_X \xi + \sigma(X, \xi). \tag{26}$$

Again, for paracontact metric (κ, μ) manifold, we get

$$\widetilde{\nabla}_X \xi = -\phi X + \phi h X. \tag{27}$$

Combining (26) and (27), we get

$$-\phi X + \phi h X = \nabla_X \xi + \sigma(X, \xi). \tag{28}$$

Since, the submanifold is invariant then $\phi X \in TM$. Now comparing the tangential and normal component, we have

$$\sigma(X, \xi) = 0 \qquad \text{and} \qquad \nabla_X \xi = -\phi X + \phi h X. \qquad \square$$

Theorem 4. *Let, M be a submanifold of a paracontact metric (κ, μ) manifold tangent to ξ. If M is invariant and recurrent then M is totally geodesic whatever be the dimension of the submanifold.*

Proof. For recurrent submanifolds we get

$$(\widetilde{\nabla}_X \sigma)(Y, Z) = A(X)\sigma(Y, Z), \tag{29}$$

where, A is a 1-form on M.
Again, by covariant differentiation we have

$$(\widetilde{\nabla}_X \sigma)(Y, Z) = \nabla^{\perp}{}_X \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z). \tag{30}$$

Combining (29) and (30) we get

$$\nabla^{\perp}{}_X \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z) = A(X)\sigma(Y, Z). \tag{31}$$

Taking $Z = \xi$ in (31) we have

$$\nabla^{\perp}{}_X \sigma(Y, \xi) - \sigma(\nabla_X Y, \xi) - \sigma(Y, \nabla_X \xi) = A(X)\sigma(Y, \xi). \tag{32}$$

Using Lemma 1, in equation (32) we have

$$\sigma(Y, \nabla_X \xi) = 0. \tag{33}$$

Again by Lemma 1 $\qquad \sigma(Y, -\phi X + \phi h X) = 0$. So, $\sigma(X, Y) = 0$.
Hence M is totally geodesic.

The converse part of the theorem is also true. The proof of the converse part is trivial. $\square$

Remark 2. All results in [15], has analogue in case of submanifolds of paracontact metric (κ, μ) manifold. All the proofs are similar to the proof of the above theorem.

6 Totally umbilical submanifolds of paracontact metric (κ, μ) manifolds

Definition 4. *A submanifold of a paracontact metric (κ, μ) manifold is called totally umbilical if it satisfies*

$$\sigma(X, Y) = g(X, Y)H. \tag{34}$$

Here σ is second fundamental form of the submanifold, g is the induced metric, H is mean curvature vector. X, Y are tangent to M [6].

In this section, we shall prove the following:

Theorem 5. *A totally umbilical invariant submanifold of a paracontact metric (κ, μ) manifold is totally geodesic whatever be the dimension of the submanifold.*

Proof. From Codazzi equation [6] we get

$$\widetilde{R}^{\perp}(X, Y)Z = g(Y, Z)\nabla^{\perp}_{X} H - g(X, Z)\nabla^{\perp}_{Y} H. \tag{35}$$

Here $\widetilde{R}$ is the curvature tensor of the ambient manifold and $\widetilde{R}^{\perp}$ is its normal part. Putting $Z = \xi$ in (35), we obtain

$$\widetilde{R}^{\perp}(X, Y)\xi = \eta(Y)\nabla^{\perp}_{X} H - \eta(X)\nabla^{\perp}_{Y} H. \tag{36}$$

Now from (6), we know

$$\widetilde{R}(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY). \tag{37}$$

Here κ and μ are constant. Since the right hand side of (37) is tangential to the submanifold, we get

$$\widetilde{R}^{\perp}(X, Y)\xi = 0. \tag{38}$$

By virtue of (36) and (38)

$$\eta(Y)\nabla^{\perp}_{X} H = \eta(X)\nabla^{\perp}_{Y} H. \tag{39}$$

Replacing X by ϕX in the above equation, we have

$$\nabla^{\perp}_{\phi X} H = 0. \tag{40}$$

Again by covariant differentiation of (34), we see that

$$\nabla^{\perp}_{W}\sigma(X, Y) - \sigma(\nabla_W X, Y) - \sigma(X, \nabla_W Y) = g(X, Y)\nabla^{\perp}_{W} H. \tag{41}$$

If the submanifold is invariant, then putting $Y = \xi$ in the above equation and using Proposition 2, equations (40) and (41) we immediately get

$$\sigma(X, -\phi W + \phi h W) = 0.$$

Hence,

$$\sigma(X, W) = 0.$$

So, the submanifold is totally geodesic. This completes the proof. □

7 Submanifolds of a paracontact metric (κ,μ) manifolds with parallel semi parallel and pseudo parallel second fundamental form

Definition 5. *The second fundamental form σ of a submanifold of a paracontact metric (κ,μ) manifold is called parallel if $\nabla\sigma$=0.*

Definition 6. *The second fundamental form σ of a submanifold of a paracontact metric (κ,μ) manifold is called semi parallel if $\widetilde{R}(X,Y).\sigma$=0.*

Definition 7. *The second fundamental form σ of a submanifold of a paracontact metric (κ,μ) manifold is called pseudo parallel if*

$$(\widetilde{R}(X,Y).\sigma)(U,V) = fQ(g,\sigma)(X,Y,U,V),$$

where f denotes a real valued function on $\widetilde{M}$.
Here Q is given by

$Q(E,T)(X,Y,Z,W) = -(X \wedge_E Y)T(Z,W) - T((X \wedge_E Y)Z,W) - T(Z,(X \wedge_E Y)W),$

$$(X \wedge_E Y)Z = E(Y,Z)X - E(X,Z)Y$$

for a (0,2) tensor E and an arbitrary tensor T.

For details about parallel and pscudo symmetric tensor, we refer [9], [10].

Submanifolds of trans-Sasakian manifolds with such properties have been studied in the paper [7]. Following the similar method of the paper [7] we obtain the following

Theorem 6. *An invariant submanifold of a paracontact metric (κ,μ) manifold is totally geodesic if and only if the second fundamental form of the submanifold is parallel, whatever be the dimension of the submanifold.*

Proof. Since σ is parallel, we have

$$(\nabla_W\sigma)(X,Y) = 0,$$

which implies

$$\nabla^{\perp}_W\sigma(X,Y) - \sigma(\nabla_W X,Y) - \sigma(X,\nabla_W Y) = 0. \tag{42}$$

Putting $Y = \xi$ in the above equation and applying Proposition 2 we obtain

$$\sigma(X,\nabla_W\xi) = 0. \tag{43}$$

So from Lemma 1 and the above equation (43) we obtain

$$\sigma(X,-\phi W + \phi hW) = 0. \tag{44}$$

Hence,

$$\sigma(X,W)=0.$$

So the submanifold is totally geodesic. The converse part is trivial. Hence the result. □

Theorem 7. *An invariant submanifold of a paracontact metric (κ,μ) manifold is totally geodesic if and only if the second fundamental form of the submanifold is semi-parallel, whatever be the dimension of the submanifold.*

Proof. Since σ is semi-parallel, we have

$$(\widetilde{R}(X,Y).\sigma)(U,V)=0,$$

which implies

$$\widetilde{R}^{\perp}(X,Y)\sigma(U,V)-\sigma(\widetilde{R}(X,Y)U,V)-\sigma(U,\widetilde{R}(X,Y)V)=0. \tag{45}$$

Putting $V=\xi=Y$ and using Lemma 1 we get from equation (45)

$$\sigma(U,\widetilde{R}(X,\xi)\xi)=0. \tag{46}$$

Now from (6), we know

$$\widetilde{R}(X,Y)\xi=\kappa(\eta(Y)X-\eta(X)Y)+\mu(\eta(Y)hX-\eta(X)hY). \tag{47}$$

Here κ and μ are constant.
The above equation implies

$$\widetilde{R}(X,\xi)\xi=\kappa\phi^2X+\mu hX. \tag{48}$$

From (46) and (48) we get

$$\sigma(X,U)=0.$$

Thus the submanifold is totally geodesic. The converse part is trivial. Hence the result. □

Theorem 8. *An invariant submanifold of a paracontact metric (κ,μ) manifold is totally geodesic if and only if the second fundamental form of the submanifold is pseudo parallel, whatever be the dimension of the submanifold.*

Proof. Since σ is pseudo parallel, we have

$$(\widetilde{R}(X,Y).\sigma)(U,V)=fQ(g,\sigma)(X,Y,U,V),$$

which implies

$$\begin{aligned}&\widetilde{R}^{\perp}(X,Y)\sigma(U,V)-\sigma(\widetilde{R}(X,Y)U,V)-\sigma(U,\widetilde{R}(X,Y)V)\\&=-f\{g(Y,\sigma(U,V))X-g(X,\sigma(U,V))Y+\sigma(g(Y,U)X-g(X,U)Y,V)\\&\quad+\sigma(U,g(Y,V)X-g(X,V)Y)\}.\end{aligned} \tag{49}$$

Putting $V = \xi = Y$ in equation (49) and applying Lemma 1 we obtain

$$\sigma(U, \widetilde{R}(X, \xi)\xi) = f\sigma(U, g(\xi, \xi)X - g(X, \xi)\xi). \tag{50}$$

Now from (6), we know

$$\widetilde{R}(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY). \tag{51}$$

Here κ and μ are constant.
The above equation implies

$$\widetilde{R}(X, \xi)\xi = \kappa\phi^2 X + \mu hX. \tag{52}$$

From (50) and (52) we get

$$\sigma(X, U) = 0.$$

So, the submanifold is totally geodesic. The converse part is trivial. □

8 Submanifolds of a paracontact metric (κ, μ) manifolds with concircular canonical field

Definition 8. *A vector field $\widetilde{V}$ is called a concircular vector field if it satisfies*

$$\widetilde{\nabla}_{\widetilde{Z}}\widetilde{V} = \widetilde{f}\widetilde{Z} \tag{53}$$

for any $\widetilde{Z}$ tangent to $\widetilde{M}$, where $\widetilde{f}$ is real valued function on $\widetilde{M}$. In particular, if $\widetilde{f} = 1$, then the concircular vector field $\widetilde{V}$ is called a concurrent vector field. Similarly, if $\widetilde{f} = 0$, then the concircular vector field $\widetilde{V}$ is called a parallel vector field.

Definition 9. *Let, M be a submanifold of $\widetilde{M}$ and also let f and V denote the restriction of $\widetilde{f}$ and $\widetilde{V}$ on M. Denote by V^T and $V^\perp$ the tangential and normal components of V,respectively. Associated with $\widetilde{V}$, we simply call V^T the canonical field and $V^\perp$ the caconical normal field of M.*

Lemma 2. *Let, $\widetilde{M}$ be an n-dimensional paracontact (κ, μ)-spaces with a concircular vector field $\widetilde{X}$ and M be a submanifold of $\widetilde{M}$. Then we have*

$$\nabla_Z X^T = fZ + A_{X^\perp}Z,$$

$$\nabla^\perp_Z X^\perp = -\sigma(X^T, Z),$$

for any Z tangent to M.

Proof. From (53) and Gauss and Weingarten formulae, we get,

$$\begin{aligned} fZ = \widetilde{\nabla}_Z X &= \widetilde{\nabla}_Z X^T + \widetilde{\nabla}_Z X^\perp \\ &= \nabla_Z X^T + \sigma(Z, X^T) + \nabla^\perp_Z X^\perp - A_{X^\perp}Z. \end{aligned}$$

Comparing the tangential part and normal part, we have the required results. □

Proposition 3. *The canonical field of M is concircular if and only if the shape operator $A_{X^\perp}$ in the direction of the canonical normal field $X^\perp$ is proportional to the identity map.*

Proof. Assume that the canonical field X^T is concircular. So,
$$\nabla_Z X^T = gZ$$
for some real valued function g on M.

From above Lemma, we get
$$fZ + A_{X^\perp}Z = gZ$$
Therefore,
$$A_{X^\perp} = (g-f)I$$
Then the shape operator $A_{X^\perp}$ in the direction of the canonical normal field $X^\perp$ is proportional to the identity map.

Conversely, let $A_{X^\perp} = hI$ for some real valued function h on M.
Now, from Lemma 2. we get,
$$\nabla_Z X^T = fZ + hZ = (f+h)Z$$
Clearly, The canonical field X^T is concircular. □

Theorem 9. *Let, $\widetilde{M}$ be a paracontact (κ,μ)-spaces with a concurrent vector field $\widetilde{X}$ and M be a submanifold of $\widetilde{M}$. The canonical field of M is concurrent iff the shape operator $A_{X^\perp} = 0$.*

Proof. Since $\widetilde{X}$ is a concurrent vector field. Hence from the definition of concurrent vector field we get,
$$Z = \widetilde{\nabla}_Z X = \widetilde{\nabla}_Z X^T + \widetilde{\nabla}_Z X^\perp = \nabla_Z X^T + \sigma(Z, X^T) + \nabla^\perp_Z X^\perp - A_{X^\perp}Z$$
Comparing the tangential and normal part we get,
$$Z = \nabla_Z X^T - A_{X^\perp}Z \tag{54}$$
$$\sigma(Z, X^T) + \nabla^\perp_Z X^\perp = 0 \tag{55}$$
Since the canonical field is concurrent then from (54), we get
$$A_{X^\perp} = 0.$$
Converse part is trivial from (54). Hence, the results follow. □

If we consider the submanifold M is totally geodesic, then from (55) we get $\nabla^\perp_Z X^\perp = 0$, i.e. the canonical normal field $X^\perp$ is parallel in the normal bundle. So we obtain the following:

Remark 3. *Let, $\widetilde{M}$ be a paracontact (κ,μ)-spaces with a concurrent vector field $\widetilde{X}$ and M be an invariant submanifold of $\widetilde{M}$. The canonical normal field $X^\perp$ is parallel in the normal bundle.*

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